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Weak Quasi-Partial Metric Spaces and Fixed Point Results

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ABSTRACT. In the present paper, we introduce a new generalized type of quasi-partial metric spaces so-called weak quasi-partial metric spaces. This concept is a generalization and unification of weak partial metric spaces and quasi-partial metric spaces. Also, some fixed point theorems in these spaces are given.

1 Introduction

In 1994, Matthews [11] defined the notion of partial metric spaces. He extended the concept of metric spaces by putting self-distances which are not necessarily equal zero. Also, he generalized Banach Contraction Theorem to partial metric spaces. After that, many fixed point theorems in partial metric spaces established (see, e.g., [1, 3, 4, 7])

In 1999, Heckmann [6] introduced the concept of weak partial metric spaces (for short WPMS), that generalized the notion of partial metric spaces. On the other hand, the concept of quasi-partial metric spaces was given by Karapinar et al. [8] and proved some fixed point theorems in these spaces. In 2016 Barakat and Zidan [5] introduced a new concept of generalized weak partial metric space (G_p^w-metric space) which is a unification between generalized partial metric space (G_p-metric) and weak partial metric spaces. Also, they proved some fixed point theorems in these spaces.

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In this paper, we introduce a new generalized type of quasi-partial metric spaces so-called weak quasi-partial metric spaces. This concept is a generalization and unification of weak partial metric spaces and quasi-partial metric spaces. Also, some fixed point theorems in these spaces are given. Now we will recall some definitions and results which needed in the sequel.

Definition 1.1 *[8] A quasi-partial metric is a function $q : X \times X \to R^+$ satisfying*
(QPM1) if $0 \leq q(x,x) = q(x,y) = q(y,y) \Longleftrightarrow x = y$,
(QPM2) $q(x,x) \leq q(y,x)$,
(QPM3) $q(x,z) + q(y,y) \leq q(x,y) + q(y,z)$ (triangle inequality),
for all $x,y,z \in X$. The pair (X,q) is called a quasi-partial metric space.

Note that, if $q(x,y) = q(y,x)$ for all $x,y \in X$, then (X,q) becomes a partial metric space. It is easy to see that for a partial metric p on X, the function $d_p : X \times X \to R^+$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{1.1}$$

is a metric on X. Analogously for a quasi-partial metric p on X, the function $d_q : X \times X \to R^+$ defined by

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y), \tag{1.2}$$

is a metric on X.

Definition 1.2 *[8] Let (X,q) be a quasi partial metric space.*
1. A sequence$\{x_n\}$ is called convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} q(x_n,x) = \lim_{n\to\infty} q(x,x_n) = \lim_{n\to\infty} q(x_n,x_n) = q(x,x)$.
2. A sequence $\{x_n\}$ is called Cauchy if $\lim_{n,m\to\infty} q(x_n,x_m)$ and $\lim_{n,m\to\infty} q(x_m,x_n)$ exists and are finite.
3. (X,q) is called complete if every Cauchy sequence $\{x_n\}$ in X is convergent to some $x \in X$ and $\lim_{n,m\to\infty} q(x_n,x_m) = \lim_{n,m\to\infty} q(x_m,x_n) = q(x,x)$.

Example 1.1 *[10] Let X be a set, and let $f : X \to [0,1)$ be an arbitrary one-to-one function. Set $q(x,y) = \max\{f(y) - f(x), 0\}$ for $x,y \in X$. Then q is a quasi-metric.*

Definition 1.3 *[6] A weak partial metric space (for short WPMS) on a nonempty set X is a function $p^w : X \times X \to \mathbb{R}^+$ (nonnegative reals) such that for all $x,y,z \in X$:*
(p_1^w) $x = y \Leftrightarrow p^w(x,x) = p^w(x,y) = p^w(y,y)$ (T_0-separation axiom),
(p_2^w) $p^w(x,y) = p^w(y,x)$ (symmetry)
(p_3^w) $p^w(x,y) \leq p^w(x,z) + p^w(z,y) - p^w(z,z)$ (modified triangular inequality).

Also Heckmann [6] shows that, if p^w is a weak partial metric on X, then for all $x,y \in X$, we have the following weak small self-distance property

$$p^w(x,y) \geq \frac{p^w(x,x) + p^w(y,y)}{2}.$$

weak small self-distance property shows that WPMS are not far from small self-distance axiom.

Remark 1.1 *[2] If p partial metric on X, then the functions $d_p, d_w : X \times X \to \mathbb{R}+$ given by*

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.3}$$

and

$$\begin{aligned} d_w(x,y) &= \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \\ &= p(x,y) - \min\{p(x,x), p(y,y)\} \end{aligned} \tag{1.4}$$

are ordinary metrics on X

In a WPMS, the convergence of a sequence, Cauchy sequence, completeness and continuity of a function are defined as PMS. To give some fixed point results on a WPMS, we need to the following Lemma,

Lemma 1.1 *[2] Let (X, p^w) be WPMS.*
(a) $\{x_n\}$ is a Cauchy sequence in (X, p^w) if and only if it is a Cauchy sequence in the metric space (X, d_w).
(b) (X, p^w) is complete if and only if (X, d_w) is complete.

Definition 1.4 *[2] Let (X, p^w) be a WPMS a sequence $\{x_n\}$ is called a p^w convergent to $x \in X$ if $\lim_{n,m\to\infty} p^w(x_n, x_m) = p^w(x,x)$. Thus if $x_n \to x$ in a WPMS (X, p^w), then for any $\epsilon > 0$, there exists $n_\epsilon \in N$ such that for all $n, m > n_\varepsilon$., we have*

$$|p^w(x_n, x_m) - p^w(x,x)| < \epsilon.$$

A point $x \in X$ is said to be limit point of the sequence $\{x_n\}$ and written $x_n \to x$.

On the other hand, Samet *etal*. [12] defined the notation of α-admissible mapping as follows.

Definition 1.5 *[12] Let F be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function. We say that F is an α-admissible mapping if, for all $x, y \in X$, we have*

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Fx, Fy) \geq 1.$$

Recently, Popescu [13] improved the notion of α-admissible mapping as follows.

Definition 1.6 *[13] Let F be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function. We say that F is an α-orbital admissible mapping if, for all $x, y \in X$, we have*

$$(F3) \quad \alpha(x, Fx) \geq 1 \quad \Longrightarrow \quad \alpha(Fx, F^2x) \geq 1.$$

Finally, Karapinar in [9] redefined the concept of triangular α-orbital admissible mapping to the following.

Definition 1.7 *[9] Let F be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function. We say that F is an triangular α-orbital admissible if F is right-α-orbital admissible and*

$$(F4)^{'} \quad \alpha(x,y) \geq 1 \quad and \quad \alpha(y,Fy) \geq 1 \Longrightarrow \quad \alpha(x,Fy) \geq 1,$$

and be triangular left-α-orbital admissible if F is α-orbital admissible and

$$(F4)^{''} \quad \alpha(Fx,x) \geq 1 \quad and \quad \alpha(x,y) \geq 1 \Longrightarrow \quad \alpha(Fx,y) \geq 1.$$

Lemma 1.2 *[9] Let $F : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(Fx_0, x_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Fx_n$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(x_m, Fx_n) \geq 1$ for all $m, n \in \mathbb{N}_0$ with $n < m$.*

In this paper, we introduce the connotation of weak quasi-partial metric spaces and some results of Karapinar *etal* [9] extended to the class of weak quasi-partial metric space.

2 Weak Quasi-Partial Metric Spaces

In this section we introduce the concept of weak quasi-partial metric spaces. Also, we give some properties in these spaces.

Definition 2.1 *A weak quasi-partial metric space on a nonempty set X is a function $\sigma : X \times X \to [0, \infty)$ such that, for all $x, y, z \in X$,*
(i) $\sigma(x,x) = \sigma(x,y) = \sigma(y,y) \Longleftrightarrow x = y$;
(ii) $\min\{\sigma(x,x), \sigma(y,y)\} \leq \sigma(x,y)$;
(ii) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y) - \sigma(z,z)$ (modified triangular inequality).

In this case (X, σ) is called a weak quasi-partial metric space.

Proposition 2.1 *For all x, y in a weak quasi-partial metric space (X, σ), we have*

$$\sigma(x,y) + \sigma(y,x) \geq \sigma(x,x) + \sigma(y,y). \tag{2.1}$$

To show the existence of weak quasi-partial metric space, we give the following example.

Example 2.1 *Let $X = [0, \infty[$. Define a mapping $\sigma : X \times X \to [0, \infty[$, , by*

$$\sigma(x,y) = \max\{x,y\} + \frac{y-x}{2}.$$

Firstly we fined that. If

$$x = y \Longrightarrow \sigma(x,x) = \sigma(x,y) = \sigma(y,y) = x,$$

$$\sigma(x,x) = \sigma(x,y) = \sigma(x,y) \Longrightarrow x = y.$$

Hence the condition (i) is satisfying in Definition 2.1.

Secondly, it's clear that the condition (ii) in Definition 2.1 is satisfied.

Finally, we need satisfy the condition (iii) as follows.

$$\begin{aligned}
\sigma(x,y) &= \max\{x,y\} + \frac{y-x}{2} \\
&\leq \max\{x,z\} + \max\{z,y\} - \max\{z,z\} + \frac{y+z-z-x}{2} \\
&= \max\{x,z\} + \frac{z-x}{2} + \max\{z,y\} + \frac{y-z}{2} - \max\{z,z\} \\
&= \sigma(x,z) + \sigma(z,y) - \sigma(z,y).
\end{aligned}$$

Hence (X,σ) is a weak quasi-partial metric space.
Indeed, if $x = 2$ and $y = 1$, then we have

$$p(x,x) = 2 > \frac{3}{2} = p(x,y)$$

. Hence, (X,σ) is neither weak partial metric space or quasi-partial metric space.

Now we obtain the following lemma without proof.

Lemma 2.1 *Let (X,σ) be a weak quasi-partial metric space. Then the function $d_\sigma : X \times X \to [0,\infty)$ given by*

$$d_\sigma(x,y) = \frac{1}{2}[\sigma(x,y) + \sigma(y,x)] - \min\{\sigma(x,x), \sigma(y,y)\}, \tag{2.2}$$

is a metric on X.

Definition 2.2 *Let (X,σ) be weak quasi-partial metric space, Then*
(i) A sequence$\{x_n\} \subseteq X$ is called converges to $x \in X$ if and only if

$$\sigma(x,x) = \lim_{n\to\infty} \sigma(x,x_n) = \lim_{n\to\infty} \sigma(x_n,x).$$

(ii) A sequence $\{x_n\} \subseteq X$ is called a Cauchy if and only if
$\lim_{n,m\to\infty} \sigma(x_n,x_m)$ and $\lim_{n,m\to\infty} \sigma(x_m,x_n)$ are exists and finite.
(iii) a weak quasi-partial metric space (X,σ) is to be complete if every Cauchy sequence $\{x_n\} \subseteq X$ converges, with respect to τ_σ, to point $x \in X$ such that

$$\sigma(x,x) = \lim_{n\to\infty} \sigma(x_n,x_m) = \lim_{n\to\infty} \sigma(x_m,x_n).$$

(iv) A mapping $F : X \to X$ is said to be converges at $x_0 \in X$ if for every $\epsilon > 0$, there exist $\delta > 0$ such that $F(B(x_0,\delta)) \subset B(F(x_0),\epsilon)$.

Lemma 2.2 *Let (X,σ) be a weak quasi-partial metric space and (X,d_σ) be the corresponding metric space. Then*
(i) A sequence $\{x_n\} \subseteq X$ is a Cauchy sequence in (X,d_σ) if and only if it is a Cauchy sequence in (X,σ).
(ii) (X,d_σ) is complete if and only if (X,σ) is complete. *(i) Let $\{x_n\}$ be a Cauchy sequence in (X,σ). Then there exists $a \in R$ such that for all $\epsilon > 0$, there is $n_0 \in N$ with $|\sigma(x_n,x_m) - a| < \frac{\epsilon}{2}$ and $|\sigma(x_m,x_n) - a| < \frac{\epsilon}{2}, \forall n,m > n_0$. Hence*

$$\begin{aligned} d_\sigma(x_n,x_m) &= \frac{1}{2}\left[\sigma(x_n,x_m) + \sigma(x_m,x_n)\right] - \min\left\{\sigma(x_n,x_n),(x_m,x_m)\right\} \\ &\leq \frac{1}{2}\left[|\sigma(x_n,x_m) - a| + |\sigma(x_m,x_m) - a|\right] + |a - \min\{\sigma(x_n,x_n),\sigma(x_m,x_m)\}| \\ &\leq \frac{1}{2}\left[\frac{\epsilon}{2} + \frac{\epsilon}{2}\right] + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\{x_n\}$ be a Cauchy sequence in (X,d_σ).
Conversely, let $\{x_n\}$ be a Cauchy sequence in (X,d_σ). Then for all $\frac{\epsilon}{2} > 0$, there is $n_0 \in N$ such that $d_\sigma(x_n,x_m) < \frac{\epsilon}{2}$, for all $n,m > n_0$. Therefor we obtain

$$\begin{aligned} \sigma(x_n,x_n) &\leq \sigma(x_n,x_{n_0}) + \sigma(x_{n_0},x_n) - \sigma(x_{n_0},x_{n_0}) \\ &\leq [\sigma(x_n,x_{n_0}) + \sigma(x_{n_0},x_n)] - \min\{\sigma(x_{n_0},xn_0),\sigma(x_n,x_n\} \\ &\leq 2d_\sigma(x_n,x_{n_0}) + \min\{\sigma(x_{n_0},xn_0),\sigma(x_n,x_n)\} \\ &\leq 2\epsilon + \sigma(x_{n_0},x_{n_0}). \end{aligned}$$

Then the sequence $\{\sigma(x_n,x_n)\}$ is bounded in R, so there exists $a \in R$ such that a subsequence $\{\sigma(x_{n_k},x_{n_k})\} \to a$ as $k \to \infty$. On the other hand, for all $n,m > n_0$, we have

$$\begin{aligned} |\sigma(x_n,x_n) - \sigma(x_m,x_m)| &\leq |\sigma(x_n,x_m) + \sigma(x_m,x_n) - 2\sigma(x_m,x_m)| \\ &\leq 2d_\sigma(x_n,x_m) < \epsilon \end{aligned}$$

Therefor the sequence $\{\sigma(x_n,x_n)\}$ is Cauchy sequence in R, then we get

$$\lim_{n\to\infty} \sigma(x_n,x_n) = a$$

From (ii) of definition (2.1.) we have

$$\sigma(x_n,x_m) - \min\{\sigma(x_n,x_n),\sigma(x_m,x_m)\} \geq 0.$$

Hence

$$\begin{aligned} |\sigma(x_n,x_m) - a| &\leq |\sigma(x_n,x_m) + \sigma(x_m,x_n) - \min\{\sigma(x_n,x_n),\sigma(x_m,x_m)\} - a| \\ &\leq |\sigma(x_n,x_m) + \sigma(x_m,x_n) - 2\min\{\sigma(x_n,x_n),\sigma(x_m,x_m)\}| + |\min\{\sigma(x_n,x_n),\sigma(x_m,x_m)\} - a| \\ &= 2d_\sigma(x_n,x_m) + |\min\{\sigma(x_n,x_n),\sigma(x_m,x_m)\} - a|. \end{aligned}$$

Then we have $\lim_{n\to\infty} \sigma(x_n,x_m) = a$. Similarly, we can get $\lim_{n\to\infty} \sigma(x_m,x_n) = a$.
Hence $\{x_n\}$ is Cauchy in sequence (X,σ).
(ii) Now, we prove that the completeness of (X,d_σ) implies completeness of (X,σ). Let $\{x_n\}$ be a Cauchy sequence in

(X,σ), then it is also a Cauchy sequence in (X,d_σ). Since (X,d_σ) is complete we deduce that there exists $x\in X$ such that

$$\lim_{n\to\infty} d_\sigma(x_n,x)=0.$$

Now, we show that

$$\lim_{n\to\infty}\sigma(x_n,x_m)=\lim_{n\to\infty}\sigma(x_m,x_n)=\lim_{n\to\infty}\sigma(x,x).$$

Since $\{x_n\}$ is Cauchy sequence in (X,σ), it is sufficient to show that

$$\lim_{n\to\infty}\sigma(x_n,x_n)=\lim_{n\to\infty}\sigma(x,x).$$

Also, $\{x_n\}$ converges to x in (X,d_σ), then for all $\epsilon>0$, there exist $n_0\in N$ such that $d_\sigma(x_n,x)<\frac{\epsilon}{2}$ for all $n>n_0$, we have

$$\begin{aligned}|\sigma(x_n,x_n)-\sigma(x,x)| &\leq |\sigma(x_n,x)+\sigma(x,x_n)-\sigma(x,x)-\sigma(x,x)|\\ &\leq |\sigma(x_n,x)+\sigma(x,x_n)-2\min\{\sigma(x,x),\sigma(x_n,x_n)\}|\\ &\leq 2d_\sigma(x_n,x)<2\left(\frac{\epsilon}{2}\right)=\epsilon.\end{aligned}$$

Then $\lim_{n\to\infty} d_\sigma(x_n,x)=\sigma(x,x)$ for all $x\in X$. Therefor (X,σ) is complete.
Finally, we prove that the completeness of (X,σ) implies completeness of (X,d_σ). Let $\{x_n\}$ be a Cauchy sequence in (X,d_σ), then it is also a Cauchy sequence in (X,σ). Since (X,σ) is complete we deduce that there exists $x\in X$ such that

$$\lim_{n\to\infty}\sigma(x_n,x_m)=\lim_{n\to\infty}\sigma(x_m,x_n)=\lim_{n\to\infty}\sigma(x_n,x_n)=\lim_{n\to\infty}\sigma(x,x).$$

Therefor

$$d_\sigma(x_n,x)=\frac{\sigma(x_n,x)+\sigma(x,x_n)}{2}-\min\{\sigma(x_n,x_n),\sigma(x,x)\},$$

by taking the limit as $n\to\infty$. Hence

$$\begin{aligned}\lim_{n\to\infty} d_\sigma(x_n,x) &= \frac{1}{2}[\lim_{n\to\infty}\sigma(x_n,x)+\lim_{n\to\infty}\sigma(x,x_n)]-\lim_{n\to\infty}\min\{\sigma(x_n,x_n),\sigma(x,x)\}\\ &= \frac{1}{2}[2\sigma(x,x)]-\sigma(x,x)=0.\end{aligned}$$

Therefor (X,d_σ) is complete.

3 Some Fixed Point Theorems

Throughout this work, let Λ be the set of all functions $\phi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:
(1) $\phi^{-1}(\{0\})=0$.
(2) $\Phi=\{\phi\in\Lambda\,|\phi$ is lower semi-continuous$\}$.

Also, assume that Ψ is the set of all continuous and non-decreasing functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the following condition:
(1) $\Psi=\{\psi\in\Lambda\,|\psi$ is continuous, non-decreasing$\}$.

In this section, we state and prove some fixed point theorems in weak quasi-partial metric spaces.

Theorem 3.1 *Let (X,σ) be complete weak quasi-partial metric space. Let $F : X \to X$ be self-mapping, Assume that there exist $\psi \in \Psi$, $\phi \in \Phi$, $L \geq 0$, and a function $\alpha : X \times X \to [0,\infty)$ such that for all $x, y \in X$*

$$\alpha(x,y)\psi(\sigma(Fx,Fy)) \leq \psi(M(x,y)) - \phi(M(x,y)) + LN(x,y), \tag{3.1}$$

Where

$$M(x,y) = \max\left\{\sigma(x,y), \sigma(x,Fx), \sigma(y,Fy), \frac{\sigma(x,Fy)+\sigma(y,Fx)}{2}\right\},$$
$$N(x,y) = \min\left\{d_\sigma(x,Fx), d_\sigma(y,Fy), d_\sigma(x,Fy), d_\sigma(y,Fx)\right\}.$$

Also, supposed that the following assertions hold.

(i) F is triangular α-orbit admissible.
(ii) There exist $x_0 \in X$ such that $\alpha(x_0,Fx_0) \geq 1$ and $\alpha(Fx_0,x_0) \geq 1$,
(iii) F is continuous on X is α-regular.
Then F has unique fixed point $u \in X$ and $\sigma(x,x) = 0$. Let $x_0 \in X$. There exists $x_1 \in X$ such that $F(x_0) = x_1$. For $x_1 \in X$ there exists $x_2 \in X$ such that $F(x_1) = x_2$. Continuing this process we obtain the sequence $\{x_n\}$ satisfying $F(x_{n-1}) = x_n$,for all $n \in N$.

If $\sigma(x_{n_0}, x_{n_0+1}) = \sigma(x_{n_0+1}, x_{n_0})$ for some $n_0 \in N$, then we have $x_{n_0} = x_{n_0+1}$, that is x_{n_0} is the fixed point. Therefore $\sigma(x_n,x_{n+1}) > 0, \forall n \in N$.
By (ii) There exist $x_0 \in X$ such that $\alpha(x_0,Fx_0) \geq 1$ and $\alpha(Fx_0,x_0) \geq 1$. From (i) we obtain

$$\alpha(x_0,x_1) = \alpha(x_0,Fx_0) \geq 1 \Longrightarrow \alpha(x_1,x_2) = \alpha(Fx_0,Fx_1) \geq 1,$$

$$\alpha(x_1,x_2) = \alpha(x_1,Fx_1) \geq 1 \Longrightarrow \alpha(x_2,x_3) = \alpha(Fx_1,Fx_2) \geq 1.$$

Continuous with this procedure, we have from get

$$\alpha(x_n,x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_{n+1},x_n) \geq 1 \forall n \in N. \tag{3.2}$$

From 3.1 and (3.2) we have

$$\begin{aligned}\psi(\sigma(x_n,x_{n+1})) &= \psi(\sigma(Fx_{n-1},Fx_n)) \\ &\leq \alpha(x_n,x_{n+1})\psi(\sigma(Fx_{n-1},Fx_n)) \\ &\leq \psi(M(x_{n-1},x_n)) - \phi(M(x_{n-1},x_n)) + LN(x_{n-1},x_n),\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\{\sigma(x_{n-1},x_n), \sigma(x_{n-1},Fx_{n-1}), \sigma(x_n,Fx_n), \frac{\sigma(x_{n-1},Fx_n)+\sigma(x_n,Fx_{n-1})}{2}\} \\ &= \max\{\sigma(x_{n-1},x_n), \sigma(x_{n-1},x_n), \sigma(x_n,x_{n+1}), \frac{\sigma(x_{n-1},x_{n+1})+\sigma(x_n,x_n)}{2}\}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}[\sigma(x_{n-1},x_{n+1})+\sigma(x_n,x_n)] &\leq \frac{1}{2}[\sigma(x_{n-1},x_n)+\sigma(x_n,x_{n+1})] \\ &\leq \max\{\sigma(x_{n-1},x_n), \sigma(x_n,x_{n+1})\}.\end{aligned}$$

Hence

$$M(x_{n-1}, x_n) = \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}. \tag{3.3}$$

$$\begin{aligned} N(x_{n-1}, x_n) &= \min\{d_\sigma(x_{n-1}, Fx_{n-1}), d_\sigma(x_n, Fx_n), d_\sigma(x_{n-1}, Fx_n), d_\sigma(x_n, Fx_{n-1})\} \\ &= 0 \end{aligned} \tag{3.4}$$

Now we will study the following cases:-

Case 1: If $M(x_{n-1}, x_n) = \sigma(x_n, x_{n+1})$, We obtain from (3.3),(3.3) and (3.4), we have

$$\psi(\sigma(x_n, x_{n+1})) \leq \psi(\sigma(x_n, x_{n+1})) - \phi(\sigma(x_n, x_{n+1})). \tag{3.5}$$

This implies that $\phi(\sigma(x_n, x_{n+1})) \leq 0$, Hence $\sigma(x_n, x_{n+1})) = 0$ which contradiction with $\phi(t) \geq 0$.

Case 2: If $M(x_{n-1}, x_n) = \sigma_m(x_{n-1}, x_n)$, We obtain from (3.3),(3.3) and (3.4), we have

$$\begin{aligned} \psi(\sigma(x_n, x_{n+1})) &\leq \psi(\sigma(x_{n-1}, x_n)) - \phi(\sigma(x_{n-1}, x_n)) \\ &\leq \psi(\sigma(x_{n-1}, x_n)) \quad \forall n \in N. \end{aligned}$$

From the non-decreasing of function ψ, we have

$$\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n).$$

Then the sequence $\{\sigma(x_n, x_{n+1})\}$ is non-increasing. So there exist $a \in [0, \infty)$ such that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = a$. If $a > 0$, then by taking the $\lim_{n\to\infty} \sup$ for both sides in inequality (3.5) we get

$$\lim_{n\to\infty} \sup \psi(\sigma(x_n, x_{n+1})) \leq \lim_{n\to\infty} \sup \psi(\sigma(x_n, x_{n+1})) - \lim_{n\to\infty} \sup \phi(\sigma(x_n, x_{n+1})).$$

By continuity of ψ and lower semi-continuity of ϕ, we obtain

$$\psi(a) \leq \psi(a) - \phi(a) \Longrightarrow \phi(a) = 0 \Longrightarrow a = 0.$$

Hence

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{3.6}$$

Similarly, we derive that

$$\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0. \tag{3.7}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in weak quasi-partial metric (X, σ). Suppose that $\{x_n\}$ is not left-Cauchy sequence in (X, σ). Then there exist $\epsilon > 0$ such that $k \in Z^+$, there is $n(k) > m(k) > k$ such that

$$\sigma(x_{n(k)}, x_{m(k)}) \geq \epsilon. \tag{3.8}$$

Further corresponding $m(k)$, we can shows $n(k)$ the smallest integer satisfy (3.8). Consequently, we have

$$\sigma(x_{n(k)-1}, x_{m(k)}) \leq \epsilon.$$

Hence

$$\begin{aligned}\epsilon \leq \sigma(x_{n(k)}, x_{m(k)}) &\leq \sigma(x_{n(k)}, x_{n(k)-1}) + \sigma(x_{n(k)-1}, x_{m(k)}) \\ &\quad -\sigma(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq \sigma(x_{n(k)}, x_{n(k)-1}) + \sigma(x_{n(k)-1}, x_{m(k)}) \\ &\leq \sigma(x_{n(k)}, x_{n(k)-1}) + \epsilon. \end{aligned} \tag{3.9}$$

By taking the limit as $k \to \infty$ for inequality (3.8) and from (3.9) we get

$$\lim_{k\to\infty} \sigma(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{3.10}$$

Also

$$\begin{aligned}\epsilon &\leq \lim_{k\to\infty} \sigma(x_{n(k)}, x_{m(k)}) \\ &\leq \lim_{k\to\infty} [\sigma(x_{n(k)}, x_{n(k)-1}) + \sigma(x_{n(k)-1}, x_{m(k)-1}) + \sigma(x_{m(k)-1}, x_{m(k)})].\end{aligned}$$

Hence

$$\epsilon \leq \sigma(x_{n(k)-1}, x_{m(k)-1}).$$

Since

$$\sigma(x_{n(k)-1}, x_{m(k)-1}) \leq \sigma(x_{n(k)-1}, x_{n(k)}) + \sigma(x_{n(k)}, x_{m(k)}) + \sigma(x_{m(k)}, x_{m(k)-1}).$$

By taking the limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)-1}) \leq 0 + \epsilon + 0. \tag{3.11}$$

From (3.10) and (3.11) we obtain

$$\lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)-1}) = \epsilon. \tag{3.12}$$

Therefor

$$\epsilon \leq \sigma(x_{n(k)}, x_{m(k)}) \leq \sigma(x_{n(k)}, x_{n(k)-1}) + \sigma(x_{n(k)-1}, x_{m(k)}).$$

By taking the limit as $k \to \infty$ and from (3.7), we get

$$\epsilon \leq \lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)}).$$

Also

$$\sigma(x_{n(k)-1}, x_{m(k)}) \leq \sigma(x_{n(k)-1}, x_{n(k)}) + \sigma(x_{n(k)}, x_{m(k)}) + \sigma(x_{m(k)}, x_{m(k)-1})$$

Taking the limit as $k \to \infty$ and from (3.6), (3.7) and (3.10), we obtain

$$\lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)}) = \epsilon.$$

Since F is triangular α-orbit admissible, from lemma 2.1 and 2.2 we drive that

$$\alpha(x_n, x_m) \geq 1 \quad and \quad \alpha(x_m, x_n) \geq 1 \quad \forall n > m \in N. \tag{3.13}$$

From (3.14) and (3.13) we have

$$\begin{aligned}\psi(\sigma(x_{n(k)}, x_{m(k)})) &= \psi(\sigma(Fx_{n(k)-1}, Fx_{m(k)-1}))\\ &\leq \alpha(x_{n(k)-1}, x_{m(k)-1})\psi(\sigma(Fx_{n(k)-1}, Fx_{m(k)-1}))\\ &\leq \psi(M(x_{n(k)-1}, x_{m(k)-1})) - \phi(M(x_{n(k)-1}, x_{m(k)-1})) + LN(x_{n(k)-1}, x_{m(k)-1}). \quad (3.14)\end{aligned} \tag{3.15}$$

Where

$$N(x_{n(k)-1}, x_{m(k)-1}) = \min\{d_\sigma(x_{n(k)-1}, x_{m(k)}), d_\sigma(x_{m(k)-1}, x_{m(k)}), d_\sigma(x_{n(k)-1}, x_{m(k)}), d_\sigma(x_{m(k)-1}, x_{n(k)})\}.$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} N(x_{n(k)-1}, x_{m(k)-1}) = 0, \tag{3.16}$$

and

$$M(x_{n(k)-1}, x_{m(k)-1}) = \max\{\sigma(x_{n(k)-1}, x_{m(k)-1}) + \sigma(x_{n(k)-1}, x_{n(k)}) + \sigma(x_{m(k)-1}, x_{m(k)}) + \frac{\sigma(x_{n(k)-1}, x_{m(k)}) + \sigma(x_{m(k)-1}, x_{n(k)})}{2}\}.$$

Therefor taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \max\{\epsilon + 0 + 0 + \lim_{k\to\infty} \frac{\sigma(x_{n(k)-1}, x_{m(k)}) + \sigma(x_{m(k)-1}, x_{n(k)})}{2}\}. \tag{3.17}$$

Since

$$\sigma(x_{n(k)-1}, x_{m(k)}) \leq \sigma(x_{n(k)-1}, x_{n(k)}) + \sigma(x_{n(k)}, x_{m(k)}),$$

then

$$\lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)}) \leq \epsilon. \tag{3.18}$$

From (3.17) and (3.18) we have

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \epsilon, \tag{3.19}$$

due to (3.14), (3.16) and (3.19), we obtain $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$. So $\phi(\epsilon) = 0$, which contradiction with $\epsilon > 0$.
Thus $\{x_n\}$ is left-Cauchy sequence in weak quasi-partial metric space (X, σ). Similarly, we can drive that the $\{x_n\}$ is right-Cauchy sequence in (X, σ). Therefor, $\{x_n\}$ is complete, then (X, d_σ) is also complete. Therefor, the sequence $\{x_n\} \to u \in X$ as $n \to \infty$, that is $d_\sigma(x_n, u) = 0$, then

$$\sigma(u, u) = \lim_{n\to\infty} \sigma(x_n, u) = \lim_{n\to\infty} \sigma(u, x_n).$$

From (3.6) and (3.7) we have

$$0 \leq \sigma(x_n, x_n) \leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_n).$$

By taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \sigma(x_n, x_n) = 0. \tag{3.20}$$

Since, $d_\sigma(x_n, u) = 0 = \frac{1}{2}[\sigma(x_n, u) + \sigma(u, x_n)] - \min\{\sigma(x_n, x_n), \sigma(u, u)\}$. Hence

$$0 = \lim_{n\to\infty} \frac{1}{2}[\sigma(x_n, u) + \sigma(u, x_n)] - \sigma(u, u) \Longrightarrow \sigma(u, u) = 0.$$

Now we need to prove u is fixed point of F. First, let F is continues, then

$$Fu = \lim_{n\to\infty} Fu_n = \lim_{n\to\infty} x_{n+1} = u.$$

So u is fixed point of F.

Second, let X be a α-regular. Hence it is α-right-regular, then there exist $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, u) \geq 1$. Assume that $\sigma(u, Fu) > 0$, then

$$\begin{aligned} \psi(\sigma(x_{n_k+1}, Fu)) &= \psi(\sigma(Fx_{n_k}, Fu)) \\ &\leq \alpha(x_{n_k}, u)\psi(\sigma(Fx_{n_k}, Fu)) \\ &\leq \psi(M(x_{n_k}, u)) - \phi(M(x_{n_k}, u)) + LN(x_{n_k}, u). \end{aligned} \tag{3.21}$$

Where

$$\begin{aligned} N(x_{n_k}, u) &= \min\{d_\sigma(x_{n_k}, x_{n_k+1}), d_\sigma(u, Fu) \\ &\quad , d_\sigma(x_{n(k)}, Fu), d_\sigma(u, x_{n(k)+1})\}. \end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{n\to\infty} N(x_{n_k}, u) = 0, \tag{3.22}$$

and

$$M(x_{n_k}, u) = \max\{\sigma(x_{n_k}, u), \sigma(x_{n_k}, x_{n_k+1}), \sigma(u, Fu), \frac{\sigma(x_{n_k}, Fu) + \sigma(u, x_{n_k+1})}{2}\}.$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{n\to\infty} M(x_{n_k}, u) = \max\{0, 0, \sigma(u, Fu), \frac{1}{2}\sigma(u, Fu)\} = \sigma(u, Fu). \tag{3.23}$$

By taking the limit as $k \to \infty$ for (3.21), taking (3.22) and (3.23) into account, we get

$$\psi(\sigma(u, Fu)) \leq \psi(\sigma(u, Fu)) - \phi(\sigma(u, Fu)) \Longrightarrow \phi(\sigma(u, Fu)) = 0.$$

Hence $\sigma(u, Fu) = 0$ and so $u = fu$, then u is fixed point of F.

If we put $L = 0$ in Theorem 3.1, we obtain the following.

Corollary 3.1 *Let (X, σ) be complete weak quasi-partial metric space and $F : X \to X$ be self-mapping, Assume that there exist $\psi \in \Psi$, $\psi \in \Phi$, $L \geq 0$, and a function $\alpha : X \times X \to [0, \infty)$ such that for all $x, y \in X$*

$$\alpha(x,y)\psi(\sigma(Fx,Fy)) \leq \psi(M(x,y)) - \varphi(M(x,y)). \tag{3.24}$$

Also, suppose that the following assertions hold.
(i) There exist $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq 1$,
(ii) F is α-admissible.
Then F has unique fixed point.

If we put $\psi(t) = 0$ and $\varphi(t) = (1-k)t$ in Theorem 3.1, we obtain the following Corollary.

Corollary 3.2 *Let (X, σ) be complete weak quasi-partial metric space and $F : X \to X$ be self-mapping. Suppose that there exist $k \in (0, 1]$, $L \geq 0$, and a function $\alpha : X \times X \to [0, \infty)$ such that for all $x, y \in X$*

$$\alpha(x,y)\sigma(Fx,Fy) \leq kM(x,y) + LN(x,y).$$

Also, suppose that the following assertions hold.
(i) F is triangular α-orbit admissible.
(ii) There exist $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq 1$ and $\alpha(Fx_0, x_0) \geq 1$,
(iii) F is continuous on X is α-regular.
Then F has unique fixed point.

Denote by $\acute{\Lambda}$ the set of functions $\lambda : [0, \infty_+) \to [0, \infty_+)$ satisfying the following hypotheses:
(1) λ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty_+)$,
(2) for every $\epsilon > 0$, we have $\int_0^\epsilon \lambda(t)dt > 0$.
We have the following result.

Corollary 3.3 *Let (X, σ) be complete weak quasi-partial metric space and $F : X \to X$ be self-mapping. Suppose that there exist $\lambda, \beta \in \acute{\Lambda}$, $L \geq 0$ and a function $\alpha : X \times X \to [0, \infty)$ such that for all $x, y \in X$*

$$\int_0^{\alpha(x,y)\sigma(Fx,Fy)} \lambda(t)dt \leq \int_0^{M(x,y)} \lambda(t)dt - \int_0^{M(x,y)} \beta(t)dt + LN(x,y).$$

Also, suppose that the following assertions hold.
(i) F is triangular α-orbit admissible.
(ii) There exist $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq 1$ and $\alpha(Fx_0, x_0) \geq 1$,
(iii) F is continuous on X is α-regular.
Then F has unique fixed point. It follows from Theorem 3.1 by taking.

$$\psi(t) = \int_0^{M(x,y)} \lambda(t)dt,$$

and

$$\phi(t) = \int_0^{M(x,y)} \beta(t)dt.$$

Hence we obtain the desired result.

Remark 3.1 *In [8, 2] The conclusion of Corollary 3.1, 3.2 and 3.3 are satisfied in quasi partial metric spaces and weak partial metric spaces.*

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