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Approximation of a Common Fixed Point of a Family of $G-$Nonexpansive Mappings in Banach Spaces with a Graph

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ABSTRACT. In this paper, we introduce an explicit iterative scheme for approximating common fixed points for a finite family of $G-$nonexpansive mappings in a uniformly convex Banach space with a directed graph. Our results extend and improve the results of [Tripak, O.: Common fixed points of G-nonexpansive mappings on Banach spaces with a graph, Fixed Point Theory Appl. doi:10.1186/s13663-016-0578-4 (2016)] and other results in the literature.

1 Introduction

Banach [4], proved the existence of unique fixed point for contraction mappings in a complete metric space. Due to its applications in mathematics and other related disciplines, Banach contraction principle has been generalized and extended in many directions. One of the recent generalization is due to Jachymiski. In 2008, Jachymiski [9], generalized the Banach contraction principle by combining the concepts from fixed point theory and graph theory.

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Applying this concept, he easily managed to prove the Kelisky-Rivlin theorem [10]. In 2012, Aleomraninejad et al.[2], presented some iterative scheme for $G-$contraction and $G-$nonexpansive mappings in a Banach space with a graph. In 2015, Tiammee et al.[25], proved Browders theorem and the convergence of Halpern iteration for a $G-$nonexpansive mapping in a Hilbert space with a directed graph. Recently, Tripak [26], proved weak and strong convergence of the Ishikawa iteration scheme to a common fixed points of two $G-$nonexpansive mappings in a Banach space with a directed graph. The Ishikawa iterative scheme, usually called the two-step iterative process, due to Ishikawa [8], is given by:

$$x_{n+1} \quad = \quad \alpha_n x_n + (1-\alpha_n)T(\beta_n x_n + (1-\beta_n T(x_n))),\ n \geq 0 \tag{1.1}$$

where the initial guess $x_0 \in C$ is taken arbitrary and the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0,1]$ such that $0 < \alpha_n \leq \beta_n < 1,\ \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. The Ishikawa iterative scheme used to approximate common fixed points of two $G-$nonexpansive mappings $S, T : C \to C$ is given by:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n S x_n,\ n \geq 0 \end{cases} \tag{1.2}$$

where x_0 is taken arbitrary and $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$.
Recall that a Banach space X is said to satisfy Opials property if the following inequality holds for any distinct elements x and y in X and for each sequence $\{x_n\}$ weakly convergent to x:

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

For a nonempty closed convex subset of a real uniformly convex Banach space X, the mappings $T_i (i = 1, 2)$ on C are said to satisfy Condition B if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that, for all $x \in C$,

$$\max\{||x - T_1 x||, ||x - T_2 x||\} \geq f(d(x, F))$$

where $F = F(T_1) \bigcap F(T_2)$ and $F(T_i)$ are the sets of fixed points of T_i. The following weak and strong convergence results were obtained by Orawan Tripak for a couple of $G-$nonexpansive mappings.

Theorem 1.1 *[26] Suppose X is uniformly convex* $\{\alpha_n\},\ \{\beta_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, \frac{1}{2})$, $T_i(i = 1, 2)$ *satisfy Condition B,* F *dominates* x_0, F *is dominated by* x_0 *and* $(x_0, z), (y_0, z), (z, x_0), (z, y_0) \in E(G)$ *for each* $z \in F$ *and arbitrary* $x_0 \in C$. *Then* $\{x_n\}$ *converges strongly to a common foxed point of* T_i.

Theorem 1.2 *[26] Suppose X is uniformly convex* $\{\alpha_n\},\ \{\beta_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, \frac{1}{2})$, *one of* $T_i(i = 1, 2)$ *is semi-compact,* F *dominates* x_0, F *is dominated by* x_0 *and* $(x_0, z), (y_0, z), (z, x_0), (z, y_0) \in E(G)$ *for each* $z \in F$ *and arbitrary* $x_0 \in C$. *Then* $\{x_n\}$ *converges strongly to a common foxed point of* T_i.

Theorem 1.3 *[26] Suppose X is uniformly convex* $\{\alpha_n\},\ \{\beta_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, \frac{1}{2})$. *If X satisfies Opial's property,* $I - T_i$ *is* $G-$*demi-closed at zero for each i,* F *dominates* x_0, F *is dominated by* x_0 *and* $(x_0, z_0), (y_0, z_0), (z_0, x_0), (z_0, y_0) \in E(G)$ *for* $z_0 \in F$ *and arbitrary* $x_0 \in C$, *then* $\{x_n\}$ *converges weakly to a common foxed point of* T_i.

On the other hand, finding common fixed points of a finite family $\{T_i\}_{i=1}^{k}$ of mappings acting on a Banach space is a problem that often arises in applied mathematics. In fact, many iterative process have been introduced for different classes of mappings with a nonempty set of common fixed points. Unfortunately, the existence results of common fixed points of a family of mappings are not known in many situations. Consequently, it is natural to consider approximation results for these classes of mappings. Therefore, it is desirable to devise a general iteration scheme which extends the Ishikawa iteration scheme (1.2) to a finite family of $G-$nonexpansive self mappings. To achieve this goal, we introduce a new iteration process for a finite family $\{T_i\}_{i=1}^{k}$ of $G-$nonexpansive self mappings as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_{k,n})x_n + \alpha_{k,n}T_k y_{k-1,n} \\ y_{k-1,n} = (1-\alpha_{k-1,n})x_n + \alpha_{k-1,n}T_{k-1}y_{k-2,n} \\ y_{k-2,n} = (1-\alpha_{k-2,n})x_n + \alpha_{k-2,n}T_{k-2}y_{k-3,n} \\ \vdots \\ y_{2,n} = (1-\alpha_{2,n})x_n + \alpha_{2,n}T_2 y_{1,n} \\ y_{1,n} = (1-\alpha_{1,n})x_n + \alpha_{1,n}T_1 x_n, \ n \geq 0 \\ y_{0,n} = x_n, \ for\ each\ n \in \mathbb{N} \end{cases} \tag{1.3}$$

for arbitrary $x_0 \in C$. Clearly, the above iteration scheme generalizes the Ishikawa iteration scheme in the sense that this considers a finite family of mappings $\{T_i\}_{i=1}^{k}$. The main purpose of this paper is to

(i) establish a necessary and sufficient condition for the convergence of the iteration scheme defined by (1.3);

(ii) prove some weak and strong convergence results of the iteration scheme (1.3) to a common fixed point of a finite family of $G-$nonexpansive mappings without assuming the Opial's condition.

2 Preliminaries

In this section, we provide and recall some definitions and lemmas which will be used in the later sections. A point $x \in X$ is called a fixed point of a self$-$mapping T on X if $x = T(x)$. The fixed point set of a mapping T will be denoted by $F(T)$.

A directed graph usually written as digraph is a pair: $G = (V(G), E(G))$, where $V(G)$ is a nonempty set called vertices of the graph G and $E(G) = \{(u,v) : u, v \in V(G)\}$ is set of order pairs called edges of the graph G. Let C be a nonempty subset of a real Banach space X and Δ be the diagonal of $C \times C$. Let G be a digraph such that the set $V(G)$ of its vertices coincide with C and $\Delta \subseteq E(G)$, i.e.,$E(G)$ contains all loops. Assume that G has no parallel Edges. If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}_{n=0}^{k}$ of vertices such that $x_0 = x,\ x_k = y$ and $(x_{i-1}, x_i) \in E(G)$, for $i = 1, 2, 3, ..., k$. A directed graph G is said to be transitive if, for any $x, y, z \in V(G)$ such that (x,y) and (y,z) are in $E(G)$, we have $(x,z) \in E(G)$.

Definition 2.1 *A self map $T : C \to C$ is said to be $G-$nonexpansive if it satisfies the conditions:*

(i) T preserves edges of G , i.e., $(x,y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$,

(ii) T non-increases weights of edges of G in the following way:

$$(x,y) \in E(G) \Rightarrow ||Tx - Ty|| \le ||x - y||.$$

Definition 2.2 *[22] Let $x_0 \in V(G)$ and A a subset of $V(G)$. We say that*

(i) A is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in A$.

(ii) A dominates x_0 if for each $x \in A$, $(x, x_0) \in E(G)$.

Definition 2.3 *Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a digraph such that $V(G) \subset C$. Then C is said to have property P if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Remark 2.4 *If G is transitive, then property P is equivalent to the property: if $\{x_n\}$ is a sequence in C with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ converging weakly to x in X, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Definition 2.5 *[22] Let C be a nonempty subset of a Banach space X and let $T : C \to X$ be a mapping. Then T is said to be $G-$demiclosed at $y \in X$ if, for any sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to $x \in C$, $\{Tx_n\}$ converges strongly to y and $(x_n, x_{n+1}) \in E(G)$ imply $Tx = y$.*

Definition 2.6 *[20] Let C be a subset of a metric space (X, d). A mapping $T : C \to C$ is semi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$ as $j \to \infty$.*

Lemma 2.7 *[28] Let X be a Banach space, and $R > 1$ be a fixed number. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$||\lambda x + (1-\lambda)y||^2 \quad \le \quad \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||)$$

for all $x, y \in B_R(0) = \{x \in X : ||x|| \le R\}$ and $\lambda \in [0,1]$.

Lemma 2.8 *[19] Let X be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $\limsup_{n\to\infty} ||x_n|| \le c$, $\limsup_{n\to\infty} ||y_n|| \le c$ and $\lim_{n\to\infty} ||\alpha_n x_n + (1-\alpha_n)y_n|| = c$ for some $c \ge 0$ then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

The following technical Lemmas are crucial in proving our results.

Lemma 2.9 *([1], pp 236-237) Let X be a uniformly convex Banach space , C be a nonempty bounded convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that, for any nonexpansive mapping $T : C \to X$, any finite many elements $\{x_i\}_{i=1}^n$ in C and any finite many nonnegative numbers $\{\lambda_i\}_{i=1}^n$ with $\sum_{i=1}^n \lambda_i = 1$, the following inequality holds:*

$$\gamma||T(\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i T x_i|| \le \max_{1\le i,j\le n} (||x_i - x_j|| - ||Tx_i - Tx_j||). \tag{2.1}$$

Lemma 2.10 *[18] Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If for any weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$, both $\{x_{n_j}\}$ and $\{x_{n_j+1}\}$ converge weakly to the same point in X, then the sequence $\{x_n\}$ is weakly convergent.*

3 Main Results

Throughout this section C denotes a nonempty subset of a real uniformly convex Banach space X with a digraph $G = (V(G), E(G))$ such that $V(G) \subset C$.

Proposition 3.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_i\}_{i=1}^{k}$ is a finite family of $G-$nonexpansive mappings on C. Let $z_0 \in F$ be such that (x_0, z_0) and (z_0, x_0) are in $E(G)$ for arbitrary $x_0 \in C$. Then, for a sequence $\{x_n\}$ generated by x_0 with iterative scheme defined by (1.3), we have $(x_n, z_0), (z_0, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n), (z_0, y_{i,n}), (y_{i,n}, z_0$ and (x_n, x_{n+1}) are in $E(G)$ for each $i = 1, 2, 3, \cdots, k$ and $n = 0, 1, 2, \cdots$.*

We proceed by induction. First we let $(x_0, z_0) \in E(G)$. Since T_1 is edge-preserving, we have $(T_1 x_0, z_0) \in E(G)$. By the convexity of $E(G)$, we get

$$\begin{aligned}(1-\alpha_{1,0})(x_0, z_0) + \alpha_{1,0}(T_1 x_0, z_0) &= ((1-\alpha_{1,0})x_0 + \alpha_{1,0} T_1 x_0, z_0) \\ &= (y_{1,0}, z_0) \in E(G).\end{aligned}$$

Since T_2 is edge-preserving, $(T_2 y_{1,0}, z_0) \in E(G)$ and again by the convexity of $E(G)$ we have

$$\begin{aligned}(1-\alpha_{2,0})(x_0, z_0) + \alpha_{2,0}(T_2 y_{1,0}, z_0) &= ((1-\alpha_{2,0})x_0 + \alpha_{2,0} T_2 y_{1,0}, z_0) \\ &= (y_{2,0}, z_0) \in E(G).\end{aligned}$$

Assume that

$$(y_{l,0}, z_0) \in E(G), \; for\ some\ l \in \{1, 2, 3, \cdots, k-2\}.$$

As T_{l+1} is edge-preserving, $(T_{l+1} y_{l,0}, z_0) \in E(G)$ and by using the convexity of $E(G)$, we get

$$\begin{aligned}(1-\alpha_{l+1,0})(x_0, z_0) + \alpha_{l+1,0}(T_{l+1} y_{l,0}, z_0) &= ((1-\alpha_{l+1,0})x_0 + \alpha_{l+1,0} T_{l+1} y_{l,0}, z_0) \\ &= (y_{l+1,0}, z_0) \in E(G).\end{aligned}$$

Thus

$$(y_{i,0}, z_0) \in E(G), \; for\ each\ i = 1, 2, 3, \cdots, k-1. \tag{3.1}$$

In particular, for $i = k-1$

$$(y_{k-1,0}, z_0) \in E(G).$$

Since T_k is edge-preserving, we have

$$(T_k y_{k-1,0}, z_0) \in E(G).$$

Using the convexity of $E(G)$, we have

$$\begin{aligned}(1-\alpha_{k,0})(x_0, z_0) + \alpha_{k,0}(T_k y_{k-1,0}, z_0) &= ((1-\alpha_{k,0})x_0 + \alpha_{k,0} T_k y_{k-1,0}, z_0) \\ &= (x_1, z_0) \in E(G).\end{aligned}$$

Thus we obtain

$$(y_{i,0}, z_0) \in E(G) \; for \; i = 1, 2, 3, \cdots, k-1 \; and \; (x_1, z_0) \in E(G). \tag{3.2}$$

By replacing (x_1, z_0) in place of (x_0, z_0) in the above process, we get

$$(y_{i,1}, z_0) \in E(G) \; for \; i = 1, 2, 3, \cdots, k-1 \; and \; (x_2, z_0) \in E(G). \tag{3.3}$$

Assume that $(x_m, z_0) \in E(G)$, for some $m \in \mathbb{N}$.
Since T_1 is edge-preserving, we have $(T_1 x_m, z_0) \in E(G)$ and by using the convexity of $E(G)$, we get

$$\begin{aligned} (1-\alpha_{1,m})(x_m, z_0) + \alpha_{1,m}(T_1 x_m, z_0) &= ((1-\alpha_{1,m})x_m + \alpha_{1,m} T_1 x_m, z_0) \\ &= (y_{1,m}, z_0) \in E(G) \end{aligned}$$

Since T_2 is edge-preserving, $(T_2 y_{1,m}, z_0) \in E(G)$ and $E(G)$ is convex, we obtain

$$\begin{aligned} (1-\alpha_{2,m})(x_m, z_0) + \alpha_{2,m}(T_2 y_{1,m}, z_0) &= ((1-\alpha_{2,m})x_m + \alpha_{2,m} T_2 y_{1,m}, z_0) \\ &= (y_{2,m}, z_0) \in E(G). \end{aligned}$$

By repeating the process, we conclude that

$$(y_{i,m}, z_0) \; and \; (x_{m+1}, z_0) \; are \; in \; E(G) \; for \; all \; i = 1, 2, 3, \cdots, k-1.$$

Continuing the process once again for (x_{m+1}, z_0), we have $(y_{i,m+1}, z_0)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$. Therefore, by induction, we conclude that (x_n, z_0) and $(y_{i,n}, z_0)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 0, 1, 2, 3, \cdots$. Using a similar argument, we can show that $(z_0, x_n), (z_0, y_{i,n})$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 0, 1, 2, \cdots$, under the assumption that $(z_0, x_0) \in E(G)$. The transitivity property of G implies that $(x_n, x_{n+1}), (x_n, y_{i,n}), (y_{i,n}, x_n)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 0, 1, 2, \cdots$. This completes the proof.

Lemma 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_i\}_{i=1}^{k}$ be a finite family of $G-$nonexpansive mappings on C. If $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some δ in $(0,1)$ and $(x_0, z_0), (z_0, x_0)$ are in $E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$, then for the sequence $\{x_n\}$ generated by (1.3), we have:*

(i) $||x_{n+1} - z_0|| \le ||x_n - z_0||, \; for \; each \; n \in \mathbb{N}.$

(ii) $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0, \; for \; each \; i = 1, 2, 3, \cdots, k.$

First we prove (i). Let $x_0 \in C$ and $z_0 \in F$ be as in the hypothesis and let $\{x_n\}$ be a sequence generated by (1.3). By Proposition 3.1, $(x_n, z_0), (z_0, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n)$ and (x_n, x_{n+1}) are in $E(G)$. By the $G-$nonexpansiveness of T_1, we have

$$\begin{aligned} ||y_{1,n} - z_0|| &= ||(1-\alpha_{1,n})x_n + \alpha_{1,n} T_1 x_n - z_0|| \\ &= ||(1-\alpha_{1,n})(x_n - z_0) + \alpha_{1,n}(T_1 x_n - z_0)|| \\ &\le (1-\alpha_{1,n})||x_n - z_0|| + \alpha_{1,n}||T_1 x_n - z_0|| \\ &\le (1-\alpha_{1,n})||x_n - z_0|| + \alpha_{1,n}||x_n - z_0|| \\ &= ||x_n - z_0|| \end{aligned}$$

Thus

$$||y_{1,n} - z_0|| \quad \leq \quad ||x_n - z_0||. \tag{3.4}$$

Again by (3.4) and using the $G-$nonexpansiveness of T_2, we have

$$\begin{aligned} ||y_{2,n} - z_0|| &= ||(1-\alpha_{2,n})x_n + \alpha_{2,n}T_2y_{1,n} - z_0|| \\ &= ||(1-\alpha_{2,n})(x_n - z_0) + \alpha_{2,n}(T_2y_{1,n} - z_0) \\ &\leq (1-\alpha_{2,n})||x_n - z_0|| + \alpha_{2,n}||T_2y_{1,n} - z_0|| \\ &\leq (1-\alpha_{2,n})||x_n - z_0|| + \alpha_{2,n}||y_{1,n} - z_0|| \\ &\leq (1-\alpha_{2,n})||x_n - z_0|| + \alpha_{2,n}||x_n - z_0|| \\ &= ||x_n - z_0|| \end{aligned}$$

which implies,

$$||y_{2,n} - z_0|| \quad \leq \quad ||x_n - z_0||. \tag{3.5}$$

Assume that

$$||y_{l,n} - z_0|| \leq ||x_n - z_0|| \tag{3.6}$$

for some $l \in \{1, 2, 3, \cdots, k-2\}$. By $G-$nonexpansiveness of T_{l+1} and (3.6), we have

$$\begin{aligned} ||y_{l+1,n} - z_0|| &= ||(1-\alpha_{l+1,n})(x_n - z_0) + \alpha_{l+1,n}(T_{l+1}y_{l,n} - z_0)|| \\ &\leq (1-\alpha_{l+1,n})||x_n - z_0|| + \alpha_{l+1,n}||T_{l+1}y_{l,n} - z_0|| \\ &\leq (1-\alpha_{l+1,n})||x_n - z_0|| + \alpha_{l+1,n}||x_n - z_0|| \\ &= ||x_n - z_0|| \end{aligned}$$

Therefore, for each $i = 1, 2, 3, \cdots, k-1$, we have

$$||y_{i,n} - z_0|| \quad \leq \quad ||x_n - z_0||. \tag{3.7}$$

In particular,

$$||y_{k-1,n} - z_0|| \leq ||x_n - z_0||,$$

so that by the $G-$nonexpansiveness of T_k we get

$$\begin{aligned} ||x_{n+1} - z_0|| &= ||(1-\alpha_{k,n})(x_n - z_0) + \alpha_{k,n}(T_ky_{k-1,n} - z_0)|| \\ &\leq (1-\alpha_{k,n})||x_n - z_0|| + \alpha_{k,n}||y_{k-1,n} - z_0|| \\ &\leq (1-\alpha_{k,n})||x_n - z_0|| + \alpha_{k,n}||x_n - z_0|| \\ &= ||x_n - z_0|| \end{aligned}$$

Therefore,

$$||x_{n+1} - z_0|| \quad \leq \quad ||x_n - z_0|| \; for \; each \; n = 0, 1, 2, \cdots. \tag{3.8}$$

Next, we prove (ii). From (i), we have $\lim_{n\to\infty} ||x_n - z_0||$ exists and hence $\{x_n - z_0\}$ is bounded. Let

$$\lim_{n\to\infty} ||x_n - z_0|| \quad = \quad c, \ for\ some\ c \geq 0. \tag{3.9}$$

If $c = 0$, then for each $l \in \{1, 2, 3, \cdots, k\}$

$$\begin{aligned} ||x_n - T_l x_n|| &\leq ||x_n - z_0|| + ||z_0 - T_l x_n|| \\ &\leq ||x_n - z_0|| + ||z_0 - x_n|| \to 0 \ as\ n \to \infty. \end{aligned}$$

Thus, for each $i = 1, 2, 3, \cdots, k$ we have

$$\lim_{n\to\infty} ||x_n - T_i x_n|| \quad = \quad 0. \tag{3.10}$$

Suppose that $c > 0$.
First, we show that

$$\lim_{n\to\infty} ||y_{l,n} - z_0|| \quad = \quad c, \tag{3.11}$$

for each $l \in \{1, 2, 3, \cdots, k-1\}$. From (3.4), we have

$$||y_{l,n} - z_0|| \quad \leq \quad ||x_n - z_0||$$

and using (3.9), we get

$$\limsup_{n\to\infty} ||y_{l,n} - z_0|| \leq c. \tag{3.12}$$

On the other hand by Lemma 2.7, we have

$$\begin{aligned} ||x_{n+1} - z_0||^2 &= ||(1-\alpha_{k,n})(x_n - z_0) + \alpha_{k,n}(T_k y_{k-1,n} - z_0)||^2 \\ &\leq (1-\alpha_{k,n})||x_n - z_0||^2 + \alpha_{k,n}||T_k y_{k-1,n} - z_0||^2 \\ &\quad -\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \\ &\leq (1-\alpha_{k,n})||x_n - z_0||^2 + \alpha_{k,n}||y_{k-1,n} - z_0||^2 \\ &\quad -\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \\ &\leq (1-\alpha_{k,n})||x_n - z_0||^2 + \alpha_{k,n}||x_n - z_0||^2 \\ &\quad -\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \\ &= ||x_n - z_0||^2 - \alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||). \end{aligned}$$

By rearranging, we obtain

$$\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \quad \leq \quad ||x_n - z_0||^2 - ||x_{n+1} - z_0||^2$$

which can be simplified to

$$\delta^2 g(||x - T_k y_{k-1,n}||) \quad \leq \quad ||x_n - z_0||^2 - ||x_{n+1} - z_0||^2. \tag{3.13}$$

Letting $n \to \infty$ in (3.13), we get

$$\lim_{n\to\infty} g(||x_n - T_k y_{k-1,n}||) = 0. \tag{3.14}$$

Since g is continuous and strictly increasing with $g(0) = 0$, we must have

$$\lim_{n\to\infty} ||x_n - T_k y_{k-1,n}|| = 0. \tag{3.15}$$

Now, using the $G-$nonexpansiveness of T_k, we have

$$\begin{aligned} ||x_n - z_0|| &\leq ||x_n - T_k y_{k-1,n}|| + ||T_k y_{k-1,n} - z_0|| \\ &\leq ||x_n - T_k y_{k-1,n}|| + ||y_{k-1,n} - z_0|| \end{aligned}$$

which implies that

$$||x_n - z_0|| \leq ||x_n - T_k y_{k-1,n}|| + ||y_{k-1,n} - z_0||. \tag{3.16}$$

But

$$\begin{aligned} ||y_{k-1,n} - z_0|| &= ||(1-\alpha_{k-1,n})(x_n - z_0) + \alpha_{k-1,n}(T_{k-1}y_{k-2,n} - z_0)|| \\ &\leq (1-\alpha_{k-1,n})||x_n - z_0|| + \alpha_{k-1,n}||T_{k-1}y_{k-2,n} - z_0|| \\ &\leq (1-\alpha_{k-1,n})||x_n - z_0|| + \alpha_{k-1,n}||y_{k-2,n} - z_0|| \end{aligned}$$

which implies that

$$||y_{k-1,n} - z_0|| \leq (1-\alpha_{k-1,n})||x_n - z_0|| + \alpha_{k-1,n}||y_{k-2,n} - z_0|| \tag{3.17}$$

Substituting (3.17) in (3.16), we get

$$\begin{aligned} ||x_n - z_0|| \leq\ & ||x_n + T_k y_{k-1,n}|| + (1-\alpha_{k-1,n})||x_n - z_0|| \\ & + \alpha_{k-1,n}||y_{k-2,n} - z_0||. \end{aligned}$$

By rearranging and making some simplifications, we obtain that

$$\begin{aligned} ||x_n - z_0|| &\leq \frac{1}{\alpha_{k-1,n}}||x_n - T_k y_{k-1,n}|| + ||y_{k-2,n} - z_0|| \\ &\leq \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| + ||y_{k-2,n} - z_0||. \end{aligned}$$

This implies that

$$||x_n - z_0|| \leq \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| + ||y_{k-2,n} - z_0||. \tag{3.18}$$

By similar procedure as we did to get (3.17), we obtain

$$||y_{k-2,n} - z_0|| \leq (1-\alpha_{k-2,n})||x_n - z_0|| + \alpha_{k-2,n}||y_{k-3,n} - z_0||$$

and substituting in (3.18), we have

$$\begin{aligned} ||x_n - z_0|| \leq\ & \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| \\ & +(1-\alpha_{k-2,n})||x_n - z_0|| + \alpha_{k-2,n}||y_{k-3,n} - z_0||. \end{aligned} \tag{3.19}$$

Again, by making some rearrangements and simplifications of (3.19), we get

$$\alpha_{k-2,n}||x_n - z_0|| \leq \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| + \alpha_{k-2,n}||y_{k-3,n} - z_0|| \tag{3.20}$$

which can be further simplified to

$$||x_n - z_0|| \leq \frac{1}{\delta^2}||x_n - T_k y_{k-1,n}|| + ||y_{k-3,n} - z_0||. \tag{3.21}$$

Continuing the process we can reach on the generalization that, for each $l \in \{1, 2, 3, \cdots, k-1\}$,

$$||x_n - z_0|| \leq \frac{1}{\delta^{k-1-l}}||x_n - T_k y_{k-1,n}|| + ||y_{l,n} - z_0||. \tag{3.22}$$

Taking $\liminf$ of (3.22) and using (3.9) and (3.15), we obtain

$$c \leq \liminf_{n\to\infty}||y_{l,n} - z_0||. \tag{3.23}$$

By combining (3.7) and (3.23), we get

$$\lim_{n\to\infty}||y_{l,n} - z_0|| = c. \tag{3.24}$$

Since

$$\begin{aligned} ||T_l y_{l-1,n} - z_0|| &\leq ||y_{l-1,n} - z_0|| \\ &\leq ||x_n - z_0||, \end{aligned} \tag{3.25}$$

we have

$$\limsup_{n\to\infty}||T_l y_{l-1,n} - z_0|| \leq c. \tag{3.26}$$

Using (3.12), (3.26) and applying Lemma 2.8, we get

$$\lim_{n\to\infty}||x_n - T_l y_{l-1,n}|| = 0, \tag{3.27}$$

for each $l \in \{2, 3, 4, \cdots, k\}$.

On the other hand by the $G-$nonexpansiveness of T_l, we have

$$\begin{aligned} ||x_n - T_l x_n|| &\leq ||x_n - T_l y_{l-1,n}|| + ||T_l y_{l-1,n} - T_l x_n|| \\ &\leq ||x_n - T_l y_{l-1,n}|| + ||y_{l-1,n} - x_n|| \\ &= ||x_n - T_l y_{l-1,n}|| + \alpha_{l-1,n}||T_{l-1} y_{l-2,n} - x_n|| \\ &\leq ||x_n - T_l y_{l-1,n}|| + (1-\delta)||T_{l-1} y_{l-2,n} - x_n|| \to 0 \; as \; n \to \infty. \end{aligned}$$

Thus, $for \; l \in \{3, 4, 5, \cdots, k\}$, we have

$$\lim_{n\to\infty}||x_n - T_l x_n|| = 0. \tag{3.28}$$

But also we have

$$\begin{aligned} ||y_{1,n} - z_0||^2 &= ||(1-\alpha_{1,n})(x_n - z_0)\alpha_{1,n}(T_1 x_n - z_0)||^2 \\ &\leq (1-\alpha_{1,n})||x_n - z_0||^2 + \alpha||T_1 x_n - z_0||^2 \\ &\quad -\alpha_{1,n}(1-\alpha_{1,n})g(||x_n - T_1 x_n||) \\ &\leq ||x_n - z_0||^2 - \alpha_{1,n}(1-\alpha_{1,n})g(||x_n - T_1 x_n||). \end{aligned}$$

After rearranging and making some simplifications, we get

$$\delta^2 g(||x_n - T_1 x_n)|| \quad \leq \quad ||x_n - z_0||^2 - ||y_{1,n} - z_0||^2. \tag{3.29}$$

Letting $n \to \infty$ in (3.29) and using (3.9) & (3.24), we get

$$\lim_{n\to\infty} g(||x_n - T_1 x_n||) \quad = \quad 0.$$

As g is continuous and strictly increasing with $g(0) = 0$, we must have

$$\lim_{n\to\infty} ||x_n - T_1 x_n|| \quad = \quad 0. \tag{3.30}$$

By using (3.27), (3.30) and $G-$nonexpansiveness of T_2, we have

$$\begin{aligned} ||x_n - T_2 x_n|| &\leq ||x_n - T_2 y_{1,n}|| + ||T_2 y_{1,n} - T_2 x_n|| \\ &\leq ||x_n - T_2 y_{1,n}|| + ||y_{1,n} - x_n|| \\ &= ||x_n - T_2 y_{1,n}|| + \alpha_{1,n}||x_n - T_1 x_n|| \\ &\leq ||x_n - T_2 y_{1,n}|| + (1-\delta)||x_n - T_1 x_n|| \\ &\quad \to 0 \ as \ n \to \infty \end{aligned}$$

which implies that

$$\lim_{n\to\infty} ||x_n - T_2 x_n|| \quad = \quad 0. \tag{3.31}$$

Therefore, combining (3.28),(3.30) & (3.31), we conclude that

$$\lim_{n\to\infty} ||x_n - T_i x_n|| \quad = \quad 0. \tag{3.32}$$

This completes the proof. We proved the following $G-$Demiclosed principle without assuming the Opial's property of the Banach space X.

Lemma 3.3 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property P. Let $\{T_i\}_{i=1}^k$ be a $G-$nonexpansive mappings on C. Then $I - T_i$ are $G-$demiclosed at 0.*

Let $\{x_n\}$ be a sequence in C such that $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup q \in C$ as $n \to \infty$ and $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$. By property P, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, q) \in E(G)$ for all $j \in \mathbb{N}$. By remark 2.4, $(x_n, q) \in E(G)$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ weakly converges in a uniformly convex Banach space X, it is bounded and hence there exists $r \geq 0$ such that $\{x_n\} \subset D =: C \cap B(0, r)$. Then D is nonempty closed convex subset of C. Thus, $T_i : D \to C$ are $G-$nonexpansive mappings. Let $l \in \{1, 2, 3, \cdots, k\}$. By Mazur's theorem (Cf. [27]), for each positive integer n, there exists a convex combination $y_n = \sum_{i=n}^{m(n)} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=n}^{m(n)} \lambda_i = 1$ such that

$$||y_n - q|| < \frac{1}{n}. \tag{3.33}$$

Since $E(G)$ is convex and $(x_i, q) \in E(G)$ for each $i \in \mathbb{N}$, we must have that

$$\begin{aligned} \sum_{i=n}^{m(n)} \lambda_i (x_i, q) &= (\sum_{i=n}^{m(n)} \lambda_i x_i, \sum_{i=n}^{m(n)} \lambda_i q) \\ &= (\sum_{i=n}^{m(n)} \lambda_i x_i, q) \\ &= (y_n, q) \in E(G). \end{aligned}$$

From $\lim_{n\to\infty} ||x_n - T_l x_n|| = 0$, we have that: for every $\epsilon > 0$, there exists a positive integer N such that

$$||x_n - T_l x_n|| < \epsilon \; for \; every \; n > N. \tag{3.34}$$

On the other hand, from Lemma 2.9 and (3.34) we have

$$\begin{aligned}
||T_l y_n - y_n|| &= ||T_l y_n - \sum_{i=n}^{m(n)} \lambda_i T_l x_i + \sum_{i=n}^{m(n)} \lambda_i T_l x_i - \sum_{i=n}^{m(n)} \lambda_i x_i|| \\
&\leq ||T_l y_n - \sum_{i=n}^{m(n)} \lambda_i T_l x_i|| + ||\sum_{i=n}^{m(n)} \lambda_i T_l x_i - \sum_{i=n}^{m(n)} \lambda_i x_i|| \\
&\leq ||T_l y_n - \sum_{i=n}^{m(n)} \lambda_i T_l x_i|| + \sum_{i=n}^{m(n)} \lambda_i ||T_l x_i - x_i|| \\
&\leq \gamma^{-1} \max_{n \leq i,j \leq m(n)} (||x_i - x_j|| - ||T_l x_i - T_l x_j||) + \epsilon \\
&\leq \gamma^{-1} \max_{n \leq i,j \leq m(n)} (||x_i - T_l x_i|| + ||x_i - T_l x_j||) + \epsilon. \qquad (3.35)
\end{aligned}$$

Therefore, from (3.33),(3.34),(3.35) and $G-$nonexpansiveness of T_l, we have

$$\begin{aligned}
||q - T_l q|| &\leq ||q - y_n|| + ||y_n - T_l y_n|| + ||T_l y_n - T_l q|| \\
&\leq 2||q - y_n|| + \gamma^{-1} \max_{n \leq i,j \leq m(n)} (||x_i - T_l x_i|| + ||x_j - T_l x_j||) + \epsilon \\
&\leq \frac{2}{n} + \gamma^{-1}(2\epsilon) + \epsilon, \; for \; n > N.
\end{aligned}$$

Taking the supremum limit as $n \to \infty$, we obtain

$$||q - T_l q \leq \gamma^{-1}(2\epsilon) + \epsilon. \tag{3.36}$$

Since γ is monotonically increasing with $\gamma(0) = 0$ and ϵ is arbitrary, we must have

$$||q - T_l q|| = 0 \tag{3.37}$$

Therefore, $q = T_i q$ for each $i \in \{1, 2, 3, \cdots, k\}$.

Theorem 3.4 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property P. Let $\{T_i\}_{i=k}^{k}$ be a finite family of $G-$nonexpansive mappings on C with the nonempty common fixed points set $F = \bigcap_{i=1}^{k} F(T_i)$. Let $x_0 \in C$ be fixed so that F dominates x_0 and F is dominated by x_0. If $\{x_n\}$ is a sequence generated by x_0 with iterative scheme (1.3) such that $\{\alpha_{i,n}\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{k}$.*

For each $z \in F$, by dominance assumption, we have $(x_0, z), (z, x_0)$ are in $E(G)$. By Lemma 3.2, we have

$$\lim_{n\to\infty} ||x_n - z|| \; exists. \tag{3.38}$$

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0, i = 1, 2, 3, \cdots, k. \tag{3.39}$$

$$\lim_{n\to\infty} ||x_n - T_i y_{i-1,n}|| = 0, i = 1, 2, 3, \cdots, k. \tag{3.40}$$

From (3.38), we see that $\{x_n\}$ is a bounded sequence in C. Since C is nonempty closed convex subset of a uniformly convex Banach space X, it is weakly compact and hence there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some point $p \in C$. It follows from (3.39)

$$\lim_{j\to\infty} ||T_i x_{n_j} - x_{n_j}|| = 0. \tag{3.41}$$

By Lemma 3.3, $I - T_i$ are $G-$demiclosed at 0 so that $p \in F$.
To complete the proof it suffices to show that $\{x_n\}$ converges weakly to p.
To this end we need to show that $\{x_n\}$ satisfies the hypothesis of Lemma 2.10.
Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $q \in C$.
By similar arguments as above q is in F.
Now for each $j \geq 1$, we have

$$x_{n_j+1} = x_{n_j} + \alpha_{k,n_j}(T_k y_{k-1,n_j} - x_{n_j}). \tag{3.42}$$

It follows from (3.40) that

$$\lim_{j\to\infty} ||T_k y_{k-1,n_j} - x_{n_j}|| = 0. \tag{3.43}$$

Since $\alpha_{k,n_j} \in [\delta, 1-\delta]$ for each $j \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$, we have

$$\lim_{j\to\infty} \alpha_{k,n_j} ||T_k y_{k-1,n_j} - x_{n_j}|| = 0. \tag{3.44}$$

Thus, from (3.42) and (3.44), we conclude that

$$weak - \lim_{j\to\infty} x_{n_j+1} = q. \tag{3.45}$$

Therefore, the sequence $\{x_n\}$ satisfies the hypothesis of Lemma 2.10 which in turn implies that $\{x_n\}$ weakly converges to q so that $p = q$. This completes the proof.

Theorem 3.5 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property SG. Let $\{T_i\}_{i=1}^k$ be a finite family of $G-$nonexpansive mappings on C with the nonempty common fixed points set $F = \bigcap_{i=1}^{k} F(T_i)$. Let $x_0 \in C$ be fixed so that F dominates x_0 and F is dominated by x_0. If at least one of the mappings in the family is semi-compact, then the iteration $\{x_n\}$ defined by (1.3) with $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^k$.*

Assume that for some $l \in \{1, 2, 3, \cdots, k\}$, T_l is semi-compact. It follows from (3.39) and (3.40) that $\{x_n\}$ is a bounded sequence in C with

$$\lim_{n\to\infty} ||x_n - T_l x_n|| = 0. \tag{3.46}$$

By the definition of semi-compactness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for some $z \in C$,

$$\lim_{j\to\infty} ||x_{n_j} - z|| = 0. \tag{3.47}$$

Since strong convergence implies weak convergence and using Remark 2.4, we have $(x_{n_j}, z) \in E(G)$. Now it is obvious that z is a fixed point of T_l. By the $G-$nonexpansiveness of T_i for each $i \in \{1, 2, 3, \cdots, k\}$, and using (3.47) and (3.48), we have

$$\begin{aligned} ||T_i z - z|| &\leq ||T_i z - T_i x_{n_j}|| + ||T_i x_{n_j} - x_{n_j}|| + ||x_{n_j} - z|| \\ &\leq 2||z - x_{n_j}|| + ||T_i x_{n_j} - x_{n_j}|| \to 0 \; as \; j \to \infty. \end{aligned} \tag{3.48}$$

Thus z is a common fixed point of the family $\{T_i\}_{i=1}^{k}$, so that $\lim_{n\to\infty} ||x_n - z||$ exists. Hence it must be the case that

$$\lim_{n\to\infty} ||x_n - z|| = 0. \tag{3.49}$$

This completes the proof.

Remark 3.6 *(i) In all our theorems, the domination assumptions made between the initial point x_0 and the set of fixed points F are crucial. But, we relaxed the domination assumption made between x_0 and z_0 and between y_0 and z_0, which is considered in some literature (like Tipak [26]).*

(ii) Property WG, for weak convergence theorems and Property SG, for strong convergence theorems are very essential aspect, while some literatures like Tipak [26] and Suparatulatorn et.al., [22] were failed to realize it.

(iii) Since Banach spaces satisfying Opial's property are very limited, relaxing this property substantially generalizes the related results in the literature.

As our concluding remark indicates, our results were not only generalize the iterative algorithm designed for two mappings to a finite family of mappings, but also solves the common problem frequently raised in connecting graph with fixed point approximations. By relaxing the Opial's property and considering finite family of the mappings, we extend and generalize many of the results in the corresponding literature.

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