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A Two-Step Seventh-Order Extended Exponential General Linear Methods for Solving IVPs in ODEs

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ABSTRACT. In this paper, a class of extended exponential general linear methods of second stage seventh order is developed. The resulting integrator is based on the variation of constant formula for initial value problems (IVPs) in Ordinary differential equations (ODEs). Stability analysis indicate that the scheme is zero stable as the parasitic root lies in the unit disk. Lastly, numerical experiment shows that the method is accurate, ecient and compete favourably with Butcher and Wright and Calvo and Palencia methods for solving problems for which exponential general linear method is appropriate.

1 Introduction

We shall be concerned with the numerical solution of systems of ordinary differential equation of the form

$$y'(t) = Wy(t)) + N(y(t)), 0 \leq t \leq T, \ Given \ \ y(0) \tag{1.1}$$

Solving ordinary differential equations numerically is,even today, still a great challenge.This applies especially to stiff differential equations and to closely related problems involving algebraic constraints (DAEs).Although the problem seems to be solved there are already highly efficient codes based on RungeKutta methods and linear

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multistep methods questions concerning methods that lie between the traditional families are still open. I will talk today about some aspects of these so-called General linear methods.

Exponential integrators were for the first time considered in the sixties and seventies of the last century.Further historical study can be seen in Minchev and Wright [15].The convergence of exponential Adams-type methods has been studied in Calvo and Palencia [5]. This class easily enables the construction of high-order schemes, although the resulting methods are only weakly stable in the sense that all parasitic roots for lie on the unit circle.
In the present paper, we are considering a class of explicit exponential integrators that combines the benefits of exponential RungeKutta and exponential AdamsBashforth methods.In it, is possible to achieve high stage order which facilitates the construction of high-order methods with favorable stability properties for the given problems.

The development of numerical integrators for this class of problems (1.1) has attracted considerable interest. The reason for this interest cannot be far fetched , because Mathematical modelling of physical situations in areas such as electricity,Biological systems can be modelled into a system of ordinary differential equation with enviable stability properties.

Most Numerical methods for solving ordinary differential equations generally fall into two main classes: linear multistep (multivalue) and Runge-Kutta (multistage) methods. Both of these classes have well known advantages and disadvantages. Butcher [1] introduced general linear methods as a unifying approach for the traditional methods to study the properties of consistency and convergence and to formulate new methods with clear advantages over the traditional methods.

A general linear method used for the numerical solution of an autonomous system of ordinary differential equations.

$$y'(x) = f(y(x)), y(x) \in \mathbb{R}^n, f(y(x)) : \mathbb{R}^n \to \mathbb{R}^n$$

Is both multistage and multivalued. We denote the internal values of step n by

$$Y_1^{[n]}, Y_2^{[n]}, \cdots, Y_s^{[n]},$$

where we define s as the number of internal stages and n is the step number
and the derivatives evaluated at the steps by

$$f(Y_1^{[n]}), f(Y_2^{[n]}), \cdots, f(Y_s^{[n]}).$$

As the start of step number n, r quantities denoted by

$$y_1^{[n-1]}, y_2^{[n-1]}, \cdots, y_r^{[n-1]}$$

are available from approximations computed in step $n-1$, with the corresponding quantities

$$y_1^{[n]}, y_2^{[n]}, \cdots, y_r^{[n]},$$

are evaluated in the step number n. For compactness of notation,we introduce vectors $Y^{[n]}$, $y^{[n-1]}$ and $y^{[n]}$ given by

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix}, N(Y) = \begin{bmatrix} N(Y_1) \\ N(Y_2) \\ \vdots \\ N(Y_s) \end{bmatrix}, y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix}, y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}.$$

If h denotes the stepsize, then the quatities imported into and evaluated in step number n are related by

$$Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]},$$

$$y^{[n]} = hBf(Y^{[n]}) + Vy^{[n-1]}, \tag{1.2}$$

The class of exponential integrators we will consider, is based on general linear methods, which use the exponential (and related matrix functions) of the approximate Jacobian within the integrator. We require that; if L=0, then the resulting method is a general linear method, which is known as the underlying general linear method; if N(u)=0, then the numerical method will supply the exact solution if the exponential and related functions are evaluated exactly.

If h represents the stepsize and the r quantities
$y_1^{[n-1]}, y_2^{[n-1]}, \cdots, y_r^{[n-1]}$, are known, then the computations performed in step number n, of an exponential general linear method, are

$$Y_i = h\sum_{j=1}^{s} a_{ij}(hL)N(Y_j) + \sum_{j=1}^{r} u_{ij}(hL)y_j^{[n-1]} \tag{1.3}$$

$$y_i^{[n]} = \sum_{j=1}^{s} b_{ij}(hL)N(Y_j) + \sum_{j=1}^{r} v_{ij}(hL)y_j^{[n-1]} \tag{1.4}$$

The coefficients of the method a_{ij}, b_{ij}, u_{ij} and $v_{ij}$are functions of the exponential and related functions. The quantities which are assumed known at the start of step number n, must be computed using some starting method when the integration begins. We can represent the exponential general linear methods in a more compact notation by introducing the vectors $Y_i, N(Y_i, y^{[n-1]}$ and $y^{[n]}$, as

An exponential general linear method can be represented in matrix form as

$$Y_i = A(hL)hN(Y_j) + U(hL)y^{[n-1]} \tag{1.5}$$

$$y^{[n]} = B(hL)hN(Y_j) + V(hL)y^{[n-1]} \tag{1.6}$$

Often it is convenient to represent the exponential general linear method in the tableau form

Table 1:Coefficients Tableau of EGLM

$A(hL)$	$U(hL)$
$B(hL)$	$V(hL)$

2 Development of Exponential General Linear Methods

The exact representation of the solution of problem (1.1) is

$$y(t_n + h) = e^{hW} y(t_n) + e^{(t_h+h)W} \int_{t_n}^{t_n+h} e^{-\tau W} N(y, \tau) d\tau \tag{2.1}$$

Exponential time differencing scheme arises from approximating $N(y(\tau), \tau)$ by a polynomial $p(\theta)$ and then integrating exactly.

The aim of this paper is to develop a new approach to the construction and numerical analysis of extended Exponential General Linear Methods (EEGLM) for solving problems (1.1). Butcher and Wright[2] constructed the practical General Linear Methods with a considerable advantages over Calvo and Palencia [3]. However, Osisiogu and Bazuaye [4] extended the internal stages to the second level. This extension enables for the construction of methods of higher order. However, this present study is concerned with the construction of a step two order seven via the new extension. This extension has not been seen anywhere in literature.

For given starting values $y_0, y_1, \cdots y_{q-1}$, the theoretical approximation y_{n+1} at time t_{n+1},$n \leq q-1$, is given by the recurrence relation or formula

$$y_{n+1} = e^{hW} y_n + h \sum_{i=1}^{s} B_i(hW) N(Y_{ni}) + h \sum_{k=1}^{q-1} V_k(hW) N(y_{n-k}) \tag{2.2}$$

The internal stages Y_{ni}, $1 \leq i \leq s$, are defined through

$$Y_{ni} = e^{c_i hW} y_n + h \sum_{j=1}^{i-1} A_{ij}(hW)(hW) N(y_{nj}) + h \sum_{k=1}^{q-1} U_{ik}(hW) N(y_{n-k}) \tag{2.3}$$

Our interest is to extend (2.2) by a higher exponential and its related matrix functions. The extended methods becomes

$$y_{n+1} = e^{hW} y_n + h \sum_{i=1}^{s} B_i(hW) N(Y_{ni}) + h \sum_{k=1}^{q-1} V_k^{(1)}(hW) N(y_{n-k}) + h^2 \sum_{k=1}^{q-1} V_k^{(2)}(hW) N'(y_{n-k}) \tag{2.4}$$

The internal stages Y_{ni}, $1 \leq i \leq s$, are defined through

$$Y_{ni} = e^{c_i hW} y_n + h \sum_{j=1}^{i-1} A_{ij}(hW) N(Y_{nj}) + h \sum_{k=1}^{q-1} U_{ik}(hW) N(y_{n-k})$$

Table 2:Coefficient Tableau of EEGLM

A_{21} $\vdots \ddots$ $A_{s1} \cdots A_{s,s-1}$	$U_{21} \cdots U_{2,q-1}$ $\vdots \cdots \vdots$ $U_{s1} \cdots U_{s,q-1}$	
$B_1 \cdots B_{s-1} B_s$	$V_1^{(1)} \cdots V_{q-1}^{(1)}$	$V_1^{(2)} \cdots V_{q-1}^{(2)}$

We assume throughout this paper that these conditions $U_{ik}(hL) = 0$ which implies $c_1 = 0$ and thus $y_{n1} = y_n$ are satisfied. The coefficients can be represented in a tablau as seen above

Before constructing methods arising from this met hod class, we derive the order conditions. Constructing step two methods in particular of different order, the order conditions is very crucial. Most times, deriving the order conditions is very difficult. This task of deriving the order condition for step two schemes has recently been handled by Bazuaye and Osisiogu.

3 Construction of Families of Two Step Method of Order Seven

The order conditions of the 723 scheme

$$y_{n+1} = e^{hW}y_n + hB_1(hW)N(Y_{n1}) + hB_2(hW)N(Y_{n2}) + hV_1^{(1)}(hL)N(y_{n-1}) + hV_2^{(1)}(hW)N(y_{n-2})$$
$$+h^2V_1^{(1)}(hW)N'(y_{n-1}) + h^2V_2^{(1)}(hW)N'(y_{n-2})$$

With $Y_{n1} = y_n$

$$Y_{n2} = e^{c_2hW}y_n + hA_{21}^{(2)}(hW)N(y_n) + hU_{21}(hW)N(y_{n-1}) + hU_{22}(hW)N(y_{n-2})$$

with $Y_{n1} = y_n$ is given as

$$c_i^\ell \eta_\ell(c_ihW) = \sum_{j=1}^{i-1} \frac{c_j^{\ell-1}}{(\ell-1)!}A_{ij}(hW) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-1}}{(\ell-1)!}U_{ik}(hW) \tag{3.1}$$

and

$$\eta_\ell(hW) = \sum_{i=1}^{s} \frac{c_i^{\ell-1}}{(\ell-1)!}B_i(hW) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-1}}{(\ell-1)!}V_k^{(1)}(hW) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-2}}{(\ell-2)!}V_k^{(2)}(hW) \tag{3.2}$$

and so by definition $c_i = 0$ for all $1 \leq i \leq s$.

Using the order conditions above, the coefficient matrix of the extended exponential general linear methods order seven step two stage order three (known as method 723) is given as

$$c_1^0A_{21} + (-1)^0U_{21} + (-2)^0U_{22} = \eta_1$$

$$A_{21} + U_{21} + U_{22} = \eta_1 \tag{3.3}$$

$$c_1^1A_{21} + \frac{(-1)^1}{1!}U_{21} + \frac{(-1)^1}{1!}U_{22} = \eta_2$$

$$-U_{21} - U_{22} = \eta_2 \tag{3.4}$$

$$\frac{c_1^2A_{21}}{2} + \frac{(-1)^2}{2!} + \frac{(-2)^2}{2!}U_{22} = \eta_3$$

$$\frac{U_{21}}{2} + 2U_{22} = \eta_3 \tag{3.5}$$

$$\frac{c_1^3A_{21}}{3!} - \frac{U_{21}}{3!} - \frac{8U_{22}}{3!} = \eta_4$$

$$-\frac{U_{21}}{6} - \frac{8U_{22}}{6} = \eta_4 \tag{3.6}$$

Solving equations (3.3) to (3.6) gives

$$A_{21} = \eta_1 + \eta_2$$

$$U_{21} = \frac{-2}{3}[2\eta_2 + \eta_3]$$

$$U_{22} = \frac{1}{2}[\eta_2 + 2\eta_3]$$

Similarly,

$$c_1^1 B_1 + c_2^1 B_2 - V_1^{(1)} - 2V_2^{(1)} + V_1^{(2)} + V_2^{(2)} = \eta_2$$

$$B_2 - V_1^{(1)} - 2V_2^{(1)} + V_1^{(2)} + V_2^{(2)} = \eta_2. \tag{3.7}$$

$$\frac{c_1^2 B_1}{2} + \frac{c_2^2 B_2}{2} + \frac{V_1^{(1)}}{2} + \frac{4V_2^{(1)}}{2} - V_1^{(2)} - 2V_2^{(2)} = \eta_3$$

$$\frac{B_2}{2} + \frac{V_1^{(1)}}{2} + \frac{4V_2^{(1)}}{2} - V_1^{(2)} - 2V_2^{(2)} = \eta_3. \tag{3.8}$$

$$\frac{c_1^3 B_1}{3!} + \frac{c_2^3 B_2}{3!} - \frac{V_1^{(1)}}{3!} - \frac{8V_2^{(1)}}{3!} + \frac{V_1^{(2)}}{2!} + \frac{4V_2^{(2)}}{2!} = \eta_4$$

$$\frac{B_2}{6} - \frac{V_1^{(1)}}{6} - \frac{8V_2^{(1)}}{6} + \frac{V_1^{(2)}}{2} + \frac{4V_2^{(2)}}{2} = \eta_4 \tag{3.9}$$

$$\frac{c_1^4 B_1}{4!} + \frac{c_2^4 B_2}{4!} + \frac{V_1^{(1)}}{4!} + \frac{16V_2^{(1)}}{4!} - \frac{V_1^{(2)}}{3!} - \frac{8V_2^{(2)}}{3!} = \eta_5$$

$$\frac{B_2}{24} + \frac{V_1^{(1)}}{24} + \frac{16V_2^{(1)}}{24} - \frac{V_1^{(2)}}{6} - \frac{8V_2^{(2)}}{6} = \eta_5 \tag{3.10}$$

$$\frac{c_1^5 B_1}{5!} + \frac{c_2^5 B_2}{5!} + \frac{V_1^{(1)}}{5!} + \frac{32V_2^{(1)}}{5!} - \frac{V_1^{(2)}}{4!} - \frac{16V_2^{(2)}}{4!} = \eta_6$$

$$\frac{B_2}{120} - \frac{V_1^{(1)}}{120} - \frac{32V_2^{(1)}}{120} + \frac{V_1^{(2)}}{24} + \frac{16V_2^{(2)}}{24} = \eta_6 \tag{3.11}$$

Solving the systems of equations (3.7) to (3.11) simultaneously, we have

$$B_2 = \frac{1}{18} 2\eta_2 + 12\eta_3 + 39\eta_4 + 72\eta_5 + 60\eta_6$$

$$v_2^1 = -\eta_2 + 2\eta_3 + \frac{9\eta_4}{2} - 12\eta_5 - 30\eta_6$$

$$v_2^1 = -\frac{10\eta_2}{9} - \frac{19\eta_3}{6} + \frac{23\eta_4}{6} + 38\eta_5 + \frac{170\eta_6}{3}$$

$$v_1^2 = -2\eta_2 - 4\eta_3 + 9\eta_4 + 48\eta_5 + 60\eta_6$$

$$v_2^2 = \frac{\eta_2}{3} - \eta_3 + \eta_4 + 12\eta_5 + 20\eta_6$$

4 Stability Consideration of the Method

To examine the stability conditions required by this method, it is expected that the stability function of the method 723 given above must lie in the unit circle

4.1 Stabilities of Order seven Step Two and Stage Order three

We recall order seven step two and stage order three (723) Scheme

$$y_{n+1} = e^{hW} y_n + hB_1(hW)N(Y_{n1}) + hB_2(hW)N(Y_{n2}) + hV_1^{(1)}(hL)N(y_{n-1}) + hV_2^{(1)}(hW)N(y_{n-2})$$
$$+h^2V_1^{(1)}(hW)N'(y_{n-1}) + h^2V_2^{(1)}(hW)N'(y_{n-2})$$

With $Y_{n1} = y_n$

$$Y_{n2} = e^{c_2hW} y_n + hA_{21}^{(2)}(hW)N(y_n) + hU_{21}(hW)N(y_{n-1}) + hU_{22}(hW)N(y_{n-2})$$

The first characteristic polynomial is given by

$$y_{n+1} = e^z y_n = 0 \; (hW = z)$$

Dividing through by y_n

$$\frac{y_{n+1}}{y_n} = \gamma_n$$

$$\gamma_n - e^z = 0 \tag{4.3}$$

The stability graph of step two order seven Scheme is shown in figure 1 below

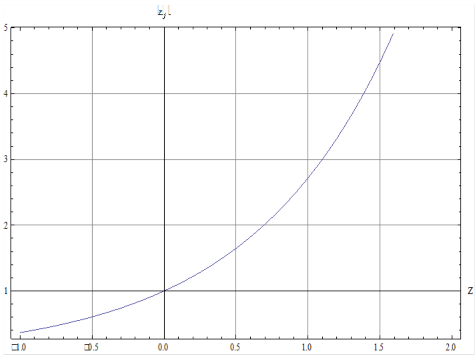

Figure 1:The stability graph of step two order seven Scheme

The graph shows by MATLAB that The stability of step two order seven Scheme is zero stable as the first characteristics polynomial lies in a unit circle

5 Discussions and Numerical Experiments

In this section, we discuss by comparing the accuracies of step two order seven exponential general linear methods with other related studies in literatures .

Problem1.

$$y' = \frac{y}{4}(1 - \frac{y}{20}) \quad y(0) = 1$$

The theoretical solution is given as

$$y = \frac{20}{1 + 19e^{\frac{-1}{4}t}}$$

The accumulated errors of the proposed scheme (723),Butcher and Wright and Calvo and Palencia for the above problem with their corresponding meshsizes are shown in the figure 2 below

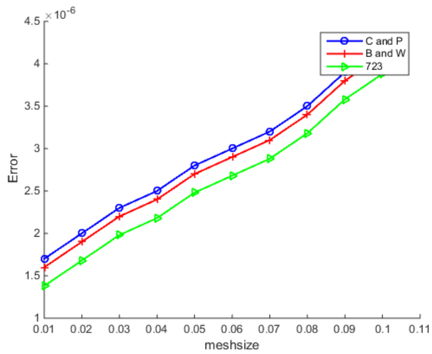

Figure 2:The relationship between the proposed method, Butcher and Wright and Calvo and Palencia

From the graph, step two order seven exhibit remarkable improvement in terms of accuracies over Butcher and Wright and Calvo and Palencia Methods.

6 Conclusion

The numerical results obtained through step two order seven scheme as indicated in figure 2, exhibit a considerable improvement over the existing methods in [2] and [3]. The numerical results presented also show that our scheme is accurate and efficient in handling the given IVP.

Finally, the stability analysis shows that the schemes are stable as the parasitic root lie in the unit disk.

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