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On Fuzzy Perfectly Disconnected Spaces

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ABSTRACT. In this paper, the concept of fuzzy perfectly disconnected space is introduced and several characterizations of fuzzy perfectly disconnected spaces, are established.

1 Introduction

In 1965, L.A. Zadeh [14] introduced the concept of fuzzy sets as a new approach for modeling uncertainities. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of Mathematics. Topology provided the most natural framework for the concepts of fuzzy sets to flourish. In 1968, C.L.Chang [5] introduced the concept of fuzzy topological spaces. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts.

Perfectly disconnected space in classical topology was defined and studied by Eric K. Van Douwen [13]. The concept of fuzzy extremally disconnected spaces was defined and studied by G.Balasubramaniam [2]. In this paper the notion of fuzzy perfectly disconnected space is introduced and several characterizations of fuzzy perfectly disconnected spaces are established. The fuzzy perfectly disconnectedness has been studied alongwith fuzzy extremally disconnectedness, fuzzy weakly Baireness and fuzzy hyper-connectedness of fuzzy topological spaces. A condition underwhich a fuzzy perfectly disconnected space becomes a fuzzy pre-Baire space is also obtained in this paper.

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2 Preliminaries

In order to make the exposition self-contained, some basic notions and results used in the sequel are given. In this work by (X,T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I, the unit interval [0,1]. A fuzzy set λ in X is a function from X into I. The null set 0 is the function from X into I which assumes only the value 0 and the whole fuzzy set 1 is the function from X into I which takes 1 only.

Definition 2.1. [5] Let (X,T) be a fuzzy topological space and λ be any fuzzy set in (X,T). The interior and the closure of λ are defined as follows:

(i) $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$

(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1-\mu \in T\}$.

Lemma 2.1. [1] Let λ be any fuzzy set in a fuzzy topological space (X,T). Then $1 - cl(\lambda) = int(1-\lambda)$ and $1 - int(\lambda) = cl(1-\lambda)$.

Definition 2.2. [8] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy dense if there exists no fuzzy closed set μ in (X,T) such that $\lambda < \mu < 1$.

Definition 2.3. [8] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu < cl(\lambda)$. That is., $int[cl(\lambda)] = 0$, in (X,T).

Definition 2.4. A fuzzy set λ in a fuzzy topological space (X,T) is called (i) fuzzy pre-open if $\lambda \leq int\ cl(\lambda)$, [7]

(ii) fuzzy pre-closed if $cl\ int\ (\lambda) \leq \lambda$, [7]

(iii) fuzzy β-open if $\lambda \leq cl\ intcl\ (\lambda)$, [3]

(iv) fuzzy β-closed if $int\ cl\ int\ (\lambda) \leq \lambda$, [3]

(v) fuzzy semi-open if $\lambda \leq cl\ int\ (\lambda)$, [1]

(vi) fuzzy semi-closed if $int\ cl\ (\lambda) \leq \lambda$, [1]

(vii) fuzzy α-open if $\lambda \leq int\ cl\ int\ (\lambda)$, [4]

$(viii)$ fuzzy α-closed if $cl\ int\ cl\ (\lambda) \leq \lambda$. [4]

Definition 2.5. [7] Let (X,T) be a fuzzy topological space. The pre-interior and the pre-closure of a fuzzy set λ are defined respectively as follows:

(i) $pint\ (\lambda) = \vee\ \{\mu\ /\ \mu \leq \lambda, \mu\ is\ a\ fuzzy\ pre-open\ set\ of\ X\}$;

(ii) $pcl\ (\lambda) = \wedge\ \{\mu/\ \lambda \leq \mu,\ \mu\ is\ a\ fuzzy\ pre-closed\ set\ of\ X\}$.

Definition 2.6. [7] A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy pre-closed set if $\lambda = pcl\ (\lambda)$ and fuzzy pre-open set if $\lambda = pint\ (\lambda)$.

Definition 2.7. [11] A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy pre-F_σ set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy pre-closed sets in (X,T).

Definition 2.8. [10] Let (X,T) be a fuzzy topological space. A fuzzy set λ in (X,T) is called a fuzzy pre-nowhere dense set if there exists no non-zero fuzzy pre-open set μ in (X,T) such that $\mu < pcl\ (\lambda)$. That is., $pint\ pcl\ (\lambda) = 0$, in (X,T).

Definition 2.9. [12] Let (X,T) be a fuzzy topological space. Then (X,T) is called a fuzzy weakly Baire space if $int(\vee_{i=1}^{\infty}(\mu_i)) = 0$, where $(\mu_i) = cl(\lambda_i) \wedge (1-\lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T).

Definition 2.10. [10] Let (X,T) be a fuzzy topological space. Then (X,T) is called a fuzzy pre-Baire space if $pint[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, where $(\lambda_i)'s$ are fuzzy pre-nowhere dense sets in (X,T).

Definition 2.11. [6] A fuzzy topological space (X,T) is said to be fuzzy hyper-connected if every non-null fuzzy open subset of (X,T) is fuzzy dense set in (X,T). That is., a fuzzy topological space (X,T) is fuzzy hyper-connected if $cl(\mu_i) = 1$, for all $\mu_i \in T$.

Definition 2.12. [2] A fuzzy topological space (X,T) is said to be fuzzy extremally disconnected if the closure of every fuzzy open set of (X,T) is fuzzy open in (X,T).

Theorem 2.1. [1]

(a) The closure of a fuzzy open set is a fuzzy regular closed set, and

(b) the interior of a fuzzy closed set is a fuzzy regular open set.

Lemma 2.2. [1] For a family $A = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \le cl\big(\vee(\lambda_\alpha)\big)$. In case A is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \le int\big(\vee(\lambda_\alpha)\big)$.

3 Fuzzy perfectly disconnected spaces

Definition 3.1. A fuzzy topological space (X,T) is called a fuzzy perfectly disconnected space if for any two non-zero fuzzy sets λ and μ defined on X with $\lambda \le 1-\mu$, $cl(\lambda) \le 1 - cl(\mu)$, in (X,T).

Example 3.1: Let $X = \{a,b,c\}$. The fuzzy sets α, β, γ, λ and μ are defined on X, as follows:

$\alpha : X \to [0,1]$ defined as $\alpha(a) = 0.4; \alpha(b) = 0.6; \alpha(c) = 0.5;$

$\beta : X \to [0,1]$ defined as $\beta(a) = 0.5; \beta(b) = 0.5, \beta(c) = 0.6;$

$\gamma : X \to [0,1]$ defined as $\gamma(a) = 0.6; \gamma(b) = 0.4; \gamma(c) = 0.5;$

$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.3; \lambda(b) = 0.5; \lambda(c) = 0.4.$

$\mu : X \to [0,1]$ defined as $\mu(a) = 0.4; \mu(b) = 0.4; \mu(c) = 0.2.$

Now $T = \{0, \alpha, \beta, \gamma, (\alpha\vee\beta), (\alpha\vee\gamma), (\beta\vee\gamma), (\alpha\wedge\beta), (\alpha\wedge\gamma), (\beta\wedge\gamma), \alpha\vee(\beta\wedge\gamma), \beta\wedge(\alpha\vee\gamma), \gamma\vee(\alpha\wedge\beta), \alpha\vee\beta\vee\gamma, 1\}$ is a fuzzy topology on X. On computations, one can see that for any two fuzzy sets δ and η in (X,T) such that $\delta \le 1-\eta, cl(\delta) \le 1 - cl(\eta)$, in (X,T). Hence (X,T) is a fuzzy perfectly disconnected space.

Example 3.2: Let $X = \{a,b,c\}$. The fuzzy sets α, β, γ, δ and η are defined on X, as follows:

$\alpha : X \to [0,1]$ defined as $\alpha(a) = 0.4; \alpha(b) = 0.6; \alpha(c) = 0.5;$

$\beta : X \to [0,1]$ defined as $\beta(a) = 0.5; \beta(b) = 0.5, \beta(c) = 0.6;$

$\gamma : X \to [0,1]$ defined as $\gamma(a) = 0.6; \gamma(b) = 0.4; \gamma(c) = 0.5;$

$\delta : X \to [0,1]$ defined as $\delta(a) = 0.5; \delta(b) = 0.4; \delta(c) = 0.6.$

$\eta : X \to [0,1]$ defined as $\eta(a) = 0.5; \eta(b) = 0.4; \eta(c) = 0.3.$

Now $T = \{0, \alpha, \beta, \gamma, (\alpha\vee\beta), (\alpha\vee\gamma), (\beta\vee\gamma), (\alpha\wedge\beta), (\alpha\wedge\gamma), (\beta\wedge\gamma), [\alpha\vee(\beta\wedge\gamma)], [\beta\wedge(\alpha\vee\gamma)], [\gamma\vee(\alpha\wedge\beta)], \alpha\vee\beta\vee\gamma, 1\}$ is a fuzzy topology on X. On computations, one can see that $\delta \le 1-\eta$ and $cl(\delta) = 1, cl(\eta) = 1 - (\alpha\vee\beta)$. Now $cl(\delta) \nleq 1 - cl(\eta)$ implies that (X,T) is not a fuzzy perfectly disconnected space.

Proposition 3.1: If (X,T) is a fuzzy perfectly disconnected space and $\lambda \le 1-\mu$ for any two non-zero fuzzy sets λ

and μ defined on X, then $cl(\lambda) \neq 1$ and $cl(\mu) \neq 1$ in (X,T).

Proof : Suppose that $\lambda \leq 1-\mu$ for any two non-zero fuzzy sets λ and μ defined on X. Since (X,T) is a fuzzy perfectly disconnected space, $cl(\lambda) \leq 1-cl(\mu)$ in (X,T). If $cl(\lambda)=1$, then $1 \leq 1-cl(\mu)$ implies that $cl(\mu) \leq 0$. That is, $cl(\mu)=0$ in (X,T). This will imply that $\mu=0$, a contradiction to $\mu \neq 0$. If $cl(\mu)=1$, then $cl(\lambda) \leq 1-1=0$. That is, $cl(\lambda)=0$. This will imply that $\lambda=0$, a contradiction to $\lambda \neq 0$. Thus, if $\lambda \leq 1-\mu$, then $cl(\lambda) \neq 1$ and $cl(\mu) \neq 1$ in (X,T).

Proposition 3.2: If $\lambda \leq 1-\mu$, and λ is a fuzzy closed set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy open set δ in (X,T) such that $\lambda \leq \delta \leq 1-\mu$.

Proof : Suppose that $\lambda \leq 1-\mu$, where λ is a fuzzy closed set in (X,T). Since (X,T) is a fuzzy perfectly disconnected space, $cl(\lambda) \leq 1-cl(\mu)$ in (X,T). Since λ is a fuzzy closed set in (X,T), $cl(\lambda)=\lambda$ and hence $\lambda \leq 1-cl(\mu)$. But $1-cl(\mu) \leq 1-\mu$, in (X,T). Therefore $\lambda \leq 1-cl(\mu) \leq 1-\mu$. Let $\delta = 1-cl(\mu)$. Then δ is a fuzzy open set in (X,T). Hence $\lambda \leq \delta \leq 1-\mu$ in (X,T), where $\delta \in T$.

Proposition 3.3: If $\lambda \leq 1-\mu$, where μ is a fuzzy closed set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy closed set η in (X,T) such that $\lambda \leq \eta \leq 1-\mu$.

Proof : Suppose that $\lambda \leq 1-\mu$, where μ is a fuzzy closed set in (X,T). Since (X,T) is a fuzzy perfectly disconnected space, $cl(\lambda) \leq 1-cl(\mu)$. Then, $cl(\lambda) \leq 1-\mu$, (since $cl(\mu)=\mu$). But $\lambda \leq cl(\lambda)$ implies that $\lambda \leq cl(\lambda) \leq 1-\mu$. Let $\eta = cl(\lambda)$. Then, η is a fuzzy closed set in (X,T). Hence $\lambda \leq \eta \leq 1-\mu$, where $1-\eta \in T$.

Proposition 3.4: If $\lambda \leq 1-\mu$, for any two fuzzy sets λ and μ defined on X, in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy open set δ in (X,T) such that $intcl(\lambda) \leq \delta \leq 1-cl[int(\mu)]$ and $int(\mu)$ is not a fuzzy dense set in (X,T).

Proof : Suppose that $\lambda \leq 1-\mu$ in (X,T). Then $cl(\lambda) \leq cl(1-\mu)$ and hence $cl(\lambda) \leq 1-int(\mu)$. Since $cl(\lambda)$ is a fuzzy closed set in (X,T), by proposition 3.2, there exists a fuzzy open set δ in (X,T) such that $cl(\lambda) \leq \delta \leq [1-int(\mu)]$. Then, $int[cl(\lambda)] \leq int(\delta) \leq int[1-int(\mu)]$ and hence $intcl(\lambda) \leq \delta \leq 1-cl[int(\mu)]$ in (X,T). Then $int\{1-cl[int(\mu)]\} \neq 0$. This implies that $1-clcl[int(\mu)] \neq 0$ and then $clint(\mu) \neq 1$. Thus $int(\mu)$ is not a fuzzy dense set in (X,T).

Proposition 3.5: If $\lambda \leq 1-\mu$, where μ is a fuzzy closed set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy regular open set δ in (X,T) such that $int(\lambda) \leq \delta \leq 1-\mu$.

Proof: Suppose that $\lambda \leq 1-\mu$, where μ is a fuzzy closed set in (X,T). Then by proposition 3.3, there exists a fuzzy closed set η in (X,T) such that $\lambda \leq \eta \leq 1-\mu$. This implies that $int(\lambda) \leq int(\eta) \leq int(1-\mu)$, in (X,T). Then, $int(\lambda) \leq int(\eta) \leq 1-cl(\mu) = 1-\mu$, in (X,T). Let $\delta = int(\eta)$. Since the interior of a fuzzy closed set is a fuzzy regular open set in a fuzzy topological space [Theorem 2.1], δ is a fuzzy regular open set in (X,T). Hence there exists a fuzzy regular open set δ in (X,T) such that $int(\lambda) \leq \delta \leq 1-\mu$.

Proposition 3.6: If $\lambda \leq 1-\mu$, where λ is a fuzzy closed set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy regular closed set η in (X,T) such that $\lambda \leq \eta \leq 1-int(\mu)$.

Proof : Suppose that $\lambda \leq 1-\mu$, where λ is a fuzzy closed set in (X,T). Then by proposition 3.2, there exists a fuzzy open set δ in (X,T) such that $\lambda \leq \delta \leq 1-\mu$. This implies that $cl(\lambda) \leq cl(\delta) \leq cl(1-\mu)$, in (X,T). Then, $\lambda \leq cl(\delta) \leq 1-int(\mu)$, in (X,T). Let $\eta = cl(\delta)$. Since the closure of a fuzzy open set is a fuzzy regular closed set in a fuzzy topological space [Theorem 2.1], η is a fuzzy regular closed set in (X,T). Hence there exists a fuzzy regular closed set η in (X, T) such that $\lambda \leq \eta \leq 1 - int(\mu)$.

Proposition 3.7: If $\lambda \leq 1 - \mu$, for any two fuzzy sets λ and μ defined on (X, T), then

(a). If μ is a fuzzy dense set in (X, T), then λ is not a fuzzy open set in (X, T).

(b). If λ is a fuzzy dense set in (X, T), then μ is not a fuzzy open set in (X, T).

(c). If λ is a fuzzy open set in (X, T), then μ is not a fuzzy dense set in (X, T).

(d). If μ is a fuzzy open set in (X, T), then λ is not a fuzzy dense set in (X, T).

(e). If $clint(\mu) = 1$, then λ is a fuzzy nowhere dense set in (X, T).

Proof:

(a). Now $\lambda \leq 1 - \mu$ implies that $clint(\lambda) \leq clint(1 - \mu)$. Then $clint(\lambda) \leq 1 - int[cl(\mu)]$ in (X, T). Since μ is a fuzzy dense set in (X, T), $cl(\mu) = 1$ and hence $intcl(\mu) = int(1) = 1$ in (X, T). Then $clint(\lambda) \leq 1 - 1 = 0$. That is, $clint(\lambda) = 0$. This implies that $int(\lambda) = 0$, in (X, T). Hence λ is not a fuzzy open set in (X, T).

(b). Since $\lambda \leq 1 - \mu$, in (X, T), $cl(\lambda) \leq 1 - int(\mu)$ in (X, T). Since λ is a fuzzy dense set in (X, T), $cl(\lambda) = 1$ and hence $1 \leq 1 - int(\mu)$. This implies that $int(\mu) = 0$ and hence μ is not a fuzzy open set in (X, T).

(c). Now $\lambda \leq 1 - \mu$, in (X, T), implies that $int(\lambda) \leq int(1 - \mu)$ and hence $int(\lambda) \leq 1 - cl(\mu)$. Since λ is a fuzzy open set, $int(\lambda) = \lambda$ in (X, T). Then $\lambda \leq 1 - cl(\mu)$. This implies that $cl(\mu) \leq 1 - \lambda$ and hence $cl(\mu) \neq 1$ in (X, T). Therefore μ is not a fuzzy dense set in (X, T).

(d). Since $\lambda \leq 1 - \mu$, $cl(\lambda) \leq cl(1 - \mu)$ and hence $cl(\lambda) \leq 1 - int(\mu)$. Since μ is a fuzzy open set in (X, T), $int(\mu) = \mu$ and thus $cl(\lambda) \leq 1 - \mu$. Hence $cl(\lambda) \neq 1$, in (X, T). Therefore λ is not a fuzzy dense set in (X, T).

(e). Now $\lambda \leq 1 - \mu$ implies that $intcl(\lambda) \leq intcl(1 - \mu)$ and then $intcl(\lambda) \leq 1 - clint(\mu)$ in (X, T). Since $clint(\mu) = 1$, $intcl(\lambda) \leq 1 - 1 = 0$. That is, $intcl(\lambda) = 0$ in (X, T). Therefore λ is a fuzzy nowhere dense set in (X, T).

Proposition 3.8: If λ is a fuzzy set in a fuzzy perfectly disconnected space (X, T), then $int(\lambda)$ is a fuzzy closed set in (X, T).

Proof: Let λ be a fuzzy set defined on X. Now, for the fuzzy set $1 - \lambda$, $1 - \lambda \leq cl(1 - \lambda)$. Then $(1 - \lambda) \leq 1 - int(\lambda)$. Since $int(\lambda)$ is a fuzzy open set in (X, T), by proposition 3.7, $1 - \lambda$ is not a fuzzy dense set in (X, T). That is, $cl(1 - \lambda) \neq 1$ in (X, T) and hence $int(\lambda) \neq 0$ in (X, T). Since (X, T) is a fuzzy perfectly disconnected space and $(1 - \lambda) \leq 1 - int(\lambda)$ in (X, T), $cl(1 - \lambda) \leq 1 - cl[int(\lambda)]$ and then $1 - int(\lambda) \leq 1 - cl[int(\lambda)]$ and thus $cl[int(\lambda)] \leq int(\lambda)$. But $int(\lambda) \leq cl[int(\lambda)]$ in (X, T) implies that $cl[int(\lambda)] = int(\lambda)$. Thus $int(\lambda)$ is a fuzzy closed set in (X, T).

Proposition 3.9: If λ is a fuzzy set in a fuzzy perfectly disconnected space (X, T), then λ is a fuzzy pre-closed set in (X, T).

Proof: Let λ be a fuzzy set defined on X in a fuzzy perfectly disconnected space (X, T). Then, by proposition 3.8, $int(\lambda)$ is a fuzzy closed set in (X, T). Let $\mu = int(\lambda)$. Since μ is a fuzzy closed set, $cl(\mu) = \mu$, in (X, T). Then, $clint(\lambda) = int(\lambda)$ and then $clint(\lambda) = int(\lambda) \leq \lambda$, in (X, T). Thus $clint(\lambda) \leq \lambda$, implies that λ is a fuzzy pre-closed set in (X, T).

Remark 3.1: In view of the above proposition, one will have the following result: "If λ is a fuzzy set in a fuzzy perfectly disconnected space (X, T), then $1 - \lambda$ is a fuzzy pre-open set in (X, T)".

Remark 3.2: In a fuzzy topological space (X, T), (i). the non-zero fuzzy open sets $(\lambda_i)'s$ are not fuzzy nowhere dense sets, since $\lambda = int(\lambda) \leq intcl(\lambda)$ and thus $intcl(\lambda) \neq 0$ in (X, T),

(ii). the fuzzy regular open sets $(\lambda_i)'s$ are not fuzzy nowhere dense sets since $intcl(\lambda) = \lambda$ and thus $intcl(\lambda) \neq 0$ in (X, T).

Proposition 3.10: If λ is a fuzzy open set in a fuzzy perfectly disconnected space (X, T), then λ is not a fuzzy nowhere dense set in (X, T).

Proof: The proof follows from Remark 3.2.

Proposition 3.11: If λ is a fuzzy regular open set in a fuzzy perfectly disconnected space (X, T), then λ is not a fuzzy nowhere dense set in (X, T).

Proof: The proof follows from Remark 3.2.

Proposition 3.12: If λ is not a fuzzy nowhere dense set in a fuzzy perfectly disconnected space (X, T), then $int[cl(\lambda)]$ is a fuzzy closed set in (X, T).

Proof: Suppose that λ is not a fuzzy nowhere dense set in (X, T). Then, $intcl(\lambda) \neq 0$, in (X, T). Now $intcl(\lambda) \leq cl(\lambda)$ in (X, T) and then $intcl(\lambda) \leq 1 - [1 - cl(\lambda)]$ in (X, T). By the fuzzy perfectly disconnectedness of (X, T), we have $cl[intcl(\lambda)] \leq 1 - cl[1 - cl(\lambda)]$ and then, $cl[intcl(\lambda)] \leq 1 - [1 - intcl(\lambda)]$ in (X, T). This implies that $cl[intcl(\lambda)] \leq int[cl(\lambda)]$. But $int[cl(\lambda)] \leq cl[intcl(\lambda)]$, in (X, T) and thus $cl[intcl(\lambda)] = intcl(\lambda)$. Then, $intcl(\lambda)$ is a fuzzy closed set in (X, T).

Proposition 3.13: If λ is a fuzzy open set in a fuzzy perfectly disconnected space (X, T), then $intcl(\lambda)$ is a fuzzy closed set in (X, T).

Proof: Let λ be a non-zero fuzzy open set in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, by proposition 3.10, λ is not a fuzzy nowhere dense set in (X, T). Then, by proposition 3.12, $intcl(\lambda)$ is a fuzzy closed set in (X, T).

Proposition 3.14: If λ is a fuzzy regular open set in a fuzzy perfectly disconnected space (X, T), then λ is a fuzzy closed set in (X, T).

Proof: Let λ be a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, by proposition 3.11, λ is not a fuzzy nowhere dense set in (X, T) and hence by proposition 3.13, $intcl(\lambda)$ is a fuzzy closed set in (X, T). Since λ is a fuzzy regular open set in (X, T), $intcl(\lambda) = \lambda$ and hence λ is a fuzzy closed set in (X, T).

Proposition 3.15: If λ is a fuzzy regular closed set in a fuzzy perfectly disconnected space (X, T), then λ is a fuzzy open set in (X, T).

Proof: Let λ be a fuzzy regular closed set in (X, T). Then, $1 - \lambda$ is a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, by proposition 3.14, $1 - \lambda$ is a fuzzy closed set in (X, T). Therefore λ is a fuzzy open set in (X, T).

Proposition 3.16: If $\lambda = [\vee_{i=1}^{\infty}(\lambda_i)]$, where $(\lambda_i)'s$ are fuzzy sets defined on a fuzzy perfectly disconnected space (X, T), then λ is a fuzzy pre-F_σ set in (X, T).

Proof: Let $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $(\lambda_i)'s$ are fuzzy sets defined on the fuzzy perfectly disconnected space (X, T). Then, by proposition 3.9, $(\lambda_i)'s$ are fuzzy pre-closed sets in (X, T) and hence λ is a fuzzy pre-F_σ set in (X, T).

Proposition 3.17: If (X, T) is a fuzzy perfectly disconnected space, then

(i). $clint(\lambda) + cl(1 - \lambda) \leq 1$, for a fuzzy set λ defined on X.

(ii). For any two fuzzy sets λ and μ with $\lambda + \mu \leq 1$, $cl(\lambda) + cl(\mu) \leq 1$.

(iii). For any two fuzzy sets λ and μ with $\lambda + \mu \leq 1$, $clint(\lambda) + clint(\mu) \leq 1$.

Proof: (i). Let λ be a fuzzy set defined on X. Then, $int(\lambda) \leq \lambda$ and thus $int(\lambda) \leq 1 - (1 - \lambda)$, in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, $clint(\lambda) \leq 1 - cl(1 - \lambda)$, in (X, T) and hence $clint(\lambda) + cl(1 - \lambda) \leq 1$.

(ii). Suppose that $\lambda + \mu \leq 1$, for any two fuzzy sets λ and μ defined on X. Then $\lambda \leq 1 - \mu$, in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T) and hence $cl(\lambda) + cl(\mu) \leq 1$.

(iii). Suppose that $\lambda + \mu \leq 1$, for any two fuzzy sets λ and μ defined on X. Then $\lambda \leq 1 - \mu$, in (X, T). Then $int(\lambda) \leq 1 - int(\mu)$. Since (X, T) is a fuzzy perfectly disconnected space, $clint(\lambda) \leq 1 - clint(\mu)$, in (X, T) and hence $clint(\lambda) + clint(\mu) \leq 1$.

In fuzzy perfectly disconnected space, fuzzy open sets are not fuzzy dense sets. For, consider the following proposition.

Proposition 3.18: If $\lambda(\neq 1)$ is a fuzzy open set in a fuzzy perfectly disconnected space (X, T), then λ is not a fuzzy dense set in (X, T).

Proof: Let λ be a fuzzy open set in (X, T). Now by proposition 3.17, $clint(\lambda) + cl(1 - \lambda) \leq 1$ in (X, T). Then, $clint(\lambda) + 1 - int(\lambda) \leq 1$. This implies that $clint(\lambda) \leq int(\lambda)$. But $int(\lambda) \leq clint(\lambda)$ and hence $cl[int(\lambda)] = int(\lambda)$. Since λ is a fuzzy open set, $int(\lambda) = \lambda$ and then $cl(\lambda) = \lambda \neq 1$. Hence λ is not a fuzzy dense set in (X, T).

Proposition 3.19: If (X, T) is a fuzzy perfectly disconnected space, then

(i). A fuzzy semi-open set in (X, T) is a fuzzy pre-open set in (X, T).

(ii). A fuzzy β-open set in (X, T) is a fuzzy pre-open set in (X, T).

(iii). The closure of a fuzzy β-open set in (X, T) is a fuzzy open set in (X, T).

(iv). $clint(\lambda) \leq intcl(\lambda)$, for a fuzzy set λ defined on X.

(v). For a fuzzy subset λ defined on X, λ is a fuzzy α-open set in (X, T) if and only if λ is a fuzzy semi-open set in (X, T).

(vi). The closure of a fuzzy semi-open set in (X, T) is a fuzzy open set in (X, T).

(vii). The closure of a fuzzy pre-open set in (X, T) is a fuzzy open set in (X, T).

Proof:

(i). Let λ be a fuzzy semi-open set in (X, T). Then, $\lambda \leq clint(\lambda)$. Now $\lambda \leq 1 - [1 - clint(\lambda)]$, in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, $cl(\lambda) \leq 1 - cl[1 - clint(\lambda)]$ in (X, T) and hence $cl(\lambda) \leq int[clint(\lambda)]$. Then, $\lambda \leq cl(\lambda) \leq int[clint(\lambda)] \leq intcl(\lambda)$ and hence $\lambda \leq intcl(\lambda)$, in (X, T). Therefore λ is a fuzzy pre-open set in (X, T).

(ii). Let λ be a fuzzy β-open set in (X, T). Then, $\lambda \leq clint[cl(\lambda)]$. Now $\lambda \leq 1 - [1 - clintcl(\lambda)]$, in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, $cl(\lambda) \leq 1 - cl[1 - clintcl(\lambda)]$ in (X, T) and hence $cl(\lambda) \leq intcl[intcl(\lambda)]$. Then, $\lambda \leq cl(\lambda) \leq intclintcl(\lambda) \leq intcl[cl(\lambda)] = intcl(\lambda)$ and hence $\lambda \leq intcl(\lambda)$, in (X, T). Therefore λ is a fuzzy pre-open set in (X, T).

(iii). Let λ be a fuzzy β-open set in (X, T). Then, by (ii), λ is a fuzzy pre-open set in (X, T) and hence $\lambda \leq intcl(\lambda)$, in (X, T). Then $cl(\lambda) \leq clint[cl(\lambda)]$. This implies that $cl(\lambda)$ is a fuzzy semi-open set in (X, T). Hence, by (i), $cl(\lambda)$ is a fuzzy pre-open set in (X, T). Then, $cl(\lambda) \leq intcl[cl(\lambda)]$ and thus $cl(\lambda) \leq int[cl(\lambda)]$. But $int[cl(\lambda)] \leq cl(\lambda)$. Then, $int[cl(\lambda)] = cl(\lambda)$. Therefore $cl(\lambda)$ is a fuzzy open set in (X, T).

(iv). Let λ be a fuzzy set defined on X. Then, by proposition 3.8, $int(\lambda)$ is a fuzzy closed set in (X,T) and hence $cl[int(\lambda)] = int(\lambda) \leq int[cl(\lambda)]$, in (X,T). Therefore $clint(\lambda) \leq intcl(\lambda)$, for a fuzzy set λ defined on X.

(v). Let λ be a fuzzy set defined on X. If λ is a fuzzy α-open set in $(X.T)$, then λ is a fuzzy semi-open set in (X,T). Conversely, suppose that λ is a fuzzy semi-open set in (X,T). Then, by (i), λ is a fuzzy pre-open set in (X,T) and hence $\lambda \leq clint(\lambda)$ and $\lambda \leq intcl(\lambda)$, in (X,T). Then $\lambda \leq intcl(\lambda) \leq intcl[clint(\lambda)] = intclint(\lambda)$. Thus $\lambda \leq intclint(\lambda)$ and therefore λ is a fuzzy α-open set in (X,T).

(vi). Let λ be a fuzzy semi-open set in (X,T). Since every fuzzy semi-open set is a fuzzy β-open set in a fuzzy topological space, λ is a fuzzy β-open set in (X,T). Then, by (iii), $cl(\lambda)$ is a fuzzy open set in (X,T).

(vii). Let λ be a fuzzy pre-open set in (X,T). Since every fuzzy pre-open set is a fuzzy β-open set in a fuzzy topological space, λ is a fuzzy β-open set in (X,T). Then, by (iii), $cl(\lambda)$ is a fuzzy open set in (X,T).

Proposition 3.20: If λ is a fuzzy nowhere dense set in a fuzzy perfectly disconnected space (X,T), then $\lambda = 0$, in (X,T).

Proof: Let λ be a fuzzy nowhere dense set in (X,T). Then, $intcl(\lambda) = 0$, in (X,T). But $int(\lambda) \leq intcl(\lambda)$, implies that $int(\lambda) = 0$, in (X,T). Since $int(1-\lambda) \leq (1-\lambda)$ in (X,T), by the fuzzy perfectly disconnectedness of (X,T), $clint(1-\lambda) \leq 1 - cl(\lambda)$. Then, $1 - intcl(\lambda) \leq 1 - cl(\lambda)$, in (X,T). This implies that $cl(\lambda) \leq intcl(\lambda)$ and hence $cl(\lambda) \leq 0$. That is, $cl(\lambda) = 0$, in (X,T). Thus, $\lambda = 0$, in (X,T).

Remark 3.3: In view of the above proposition, we have the following result: *"There is no non-zero fuzzy nowhere dense set in a fuzzy perfectly disconnected space"*.

4 Fuzzy perfectly disconnected spaces and other fuzzy topological spaces

Proposition 4.1: If (X,T) is a fuzzy perfectly disconnected space, then (X,T) is a fuzzy extremally disconnected space.

Proof: Let λ be a fuzzy set defined on X. Since (X,T) is a fuzzy perfectly disconnected space and $(1-\lambda) \leq 1 - [int(\lambda)]$ in (X,T) implies that $cl(1-\lambda) \leq 1 - cl[int(\lambda)]$ and then $1 - int(\lambda) \leq 1 - clint(\lambda)$ and hence $clint(\lambda) \leq int(\lambda)$. But $int(\lambda) \leq cl[int(\lambda)]$ in (X,T). Hence, $clint(\lambda) = int(\lambda)$, in (X,T). Let $\mu = int(\lambda)$, then μ is a fuzzy open set in (X,T). Now $cl(\mu) = \mu$ and $\mu \in T$ implies that $cl(\mu) \in T$. Hence, if $\mu \in T$ then $cl(\mu) \in T$ implies that (X,T) is a fuzzy extremally disconnected space.

Remark 4.1: This proposition can also be proved from proposition 3.17.

For, from (i) of proposition 3.17, we have $clint(\lambda) + 1 - int(\lambda) \leq 1$ and thus $clint(\lambda) \leq int(\lambda)$. Since λ is a fuzzy open set, $cl(\lambda) \leq \lambda$. But, $\lambda \leq cl(\lambda)$ in (X,T) and hence $cl(\lambda) = \lambda \in T$. Therefore (X,T) is a fuzzy perfectly disconnected space.

Proposition 4.2: If (X,T) is a fuzzy perfectly disconnected space, then (X,T) is not a fuzzy weakly Baire space.

Proof: Let (X,T) be a fuzzy perfectly disconnected space and $(\lambda_i)'s$ $(i = 1 to \infty)$ be non-zero fuzzy regular open sets in (X,T). Let $\mu_i = cl(\lambda_i) \wedge (1-\lambda_i)$. Then, $int(\mu_i) = int[cl(\lambda_i) \wedge (1-\lambda_i)] = int[cl(\lambda_i)] \wedge int[(1-\lambda_i)]$. Since the fuzzy sets $(\lambda_i)'s$ are fuzzy regular open sets in (X, T), $int[cl(\lambda_i)] = \lambda_i$ and thus $int(\mu_i) = (\lambda_i) \wedge int[(1 - \lambda_i)]$. By proposition 3.14, $(\lambda_i)'s$ are fuzzy closed sets in (X, T). Also, by proposition 3.8, $[int(1 - \lambda_i)]'s$ are fuzzy closed sets in (X, T). Then $\{\lambda_i \wedge int(1 - \lambda_i)\}'s$ are fuzzy closed sets in (X, T) and thus $[int(\mu_i)]'s$ are non-zero fuzzy closed sets in (X, T). This implies that $\vee_{i=1}^{\infty} int(\mu_i) \neq 0$. By lemma 2.2, $\vee_{i=1}^{\infty} int(\mu_i) \leq int[\vee_{i=1}^{\infty}(\mu_i)]$ and hence $int[\vee_{i=1}^{\infty}(\mu_i)] \neq 0$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T). Therefore (X, T) is not a fuzzy weakly Baire space.

Proposition 4.3: If $pint[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, where $pint(\lambda_i) = 0$, for the fuzzy sets $(\lambda_i)'s$ in a fuzzy perfectly disconnected space (X, T), then (X, T) is a fuzzy pre-Baire space.

Proof: Let $(\lambda_i)'s$ $(i = 1 to \infty)$ be fuzzy sets defined on a fuzzy perfectly disconnected space (X, T). Then, by proposition 3.9, $(\lambda_i)'s$ $(i = 1 to \infty)$ are fuzzy pre-closed sets in (X, T). Now $pint[pcl(\lambda_i)] = pint(\lambda_i) = 0$(by hypothesis). Hence $(\lambda_i)'s$ are fuzzy pre-nowhere dense sets in (X, T). Hence $pint[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, where $(\lambda_i)'s$ are fuzzy pre-nowhere dense sets in (X, T), implies that (X, T) is a fuzzy pre-Baire space.

Proposition 4.4: If (X, T) is a fuzzy perfectly disconnected space, then (X, T) is not a fuzzy hyper-connected space

Proof: Let λ be a fuzzy open set in (X, T). Then by proposition 3.18, λ is not a fuzzy dense set in (X, T). Hence (X, T) is not a fuzzy hyper-connected space.

5 Conclusion

In this paper, the concept of fuzzy perfectly disconnected spaces is defined. It is established in fuzzy perfectly disconnected spaces that the fuzzy regular open sets are fuzzy closed sets and the fuzzy semi-open and fuzzy β-open sets are fuzzy pre-open sets. Also, it is shown that the closure of fuzzy β-open, fuzzy semi-open and fuzzy pre-open sets are fuzzy open sets in fuzzy perfectly disconnected spaces and no non-zero fuzzy nowhere dense set exists in fuzzy perfectly disconnected spaces. The condition for fuzzy perfectly disconnected spaces to become a fuzzy pre-Baire spaces is also obtained and it is shown that the fuzzy perfectly disconnected spaces are fuzzy extremally disconnected spaces but not fuzzy hyper-connected spaces and not fuzzy weakly Baire spaces.

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