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Approximation of Common Solutions of Fixed Point Problem for Hemicontractive-type Mapping, Split Equilibrium and Variational Inequality Problems

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ABSTRACT. In this paper, we introduce and study an iterative approximation method for finding a common element of the set of fixed points of multi-valued hemicontractive-type mapping, the set of solutions of split equilibrium problem and the set of solutions of variational inequality problem for continuous monotone mapping in real Hilbert spaces. Furthermore, we proved strong convergence theorem for the sequences generated by the proposed iterative algorithm under some conditions imposed on parameters. Our results improve and generalize most of the results that have been previously proved by some authors recently.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Let C be a nonempty subset of

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H. A mapping $S : C \longrightarrow H$ is called *Lipschitzian* if there exists $L \geq 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \text{ for all } x, y \in C. \tag{1.1}$$

If $L = 1$ in (1.1), then S is called *nonexpansive mapping* and if $0 \leq L < 1$, then S is called a *contraction*.

A mapping $S : C \longrightarrow H$ is said to be $k-$*strictly pseudocontractive* if there exists $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|x - Sx - (y - Sy)\|^2, \text{ for all } x, y \in C.$$

And S is called *pseudocontractive* mapping if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|x - Sx - (y - Sy)\|^2, \text{ for all } x, y \in C.$$

Observe that the class of pseudocontractive mappings properly contains the classes of $k-$strictly pseudocontractive mappings and nonexpansive mappings (see [6, 15]).

A mapping $S : C \longrightarrow H$ with nonempty set of fixed points, $F(S) = \{x \in C : Sx = x\} \neq \emptyset$, is said to be *quasi-nonexpansive* if $\|Sx - p\| \leq \|x - p\|$ holds for all $p \in F(S)$ and $x \in C$. The mapping S is called *demicontractive* if there exists a constant $k \in [0, 1)$ such that $\|Sx - p\|^2 \leq \|x - p\|^2 + k\|x - Sx\|^2$, for all $p \in F(S)$ and $x \in C$. And S is called *hemicontractive* if $\|Sx - p\|^2 \leq \|x - p\|^2 + \|x - Sx\|^2$ holds for all $p \in F(S)$ and $x \in C$.

We note that the class of hemicontractive mappings properly contains the class of pseudocontractive mappings S with $F(S) \neq \emptyset$ and the classes of quasi-nonexpansive and demicontractive mappings (see, *for example,* [16, 27]).

Let $CB(C)$ denote the family of nonempty, closed and bounded subsets of C, and $K(C)$ denote the family of nonempty and compact subsets of C.

The *Hausdorff metric* on $CB(C)$ is defined by

$$D(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\},$$

for all $A, B \in CB(C)$, where $d(x, B) = \inf\{\|x - b\| : b \in B\}$.

Definition 1.1 *Let $S : C \longrightarrow CB(C)$ be a multi-valued mapping. Then, S is said to be*

i) Lipschitzian if there exists a constant $L \geq 0$ such that

$$D(Sx, Sy) \leq L\|x - y\|, \textit{ for all } x, y \in C.$$

If $L = 1$, then S is called nonexpansive;

ii) $k-$strictly pseudocontractive if there exists $k \in [0, 1)$ such that

$$D^2(Sx, Sy) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2,$$

for all $x, y \in C$, $u \in Sx$ and $v \in Sy$;

iii) pseudocontractive if

$$D^2(Sx, Sy) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2,$$

for all $x, y \in C$, $u \in Sx$ and $v \in Sy$.

We observe from the definitions that every multi-valued nonexpansive mapping is $k-$strictly pseudocontractive mapping and every multi-valued $k-$strictly pseudocontractive mapping is pseudocontractive mapping, however, the inclusions are strict (see [6, 27]).

An element $x \in C$ is called a *fixed point* of a multi-valued mapping $S : C \longrightarrow CB(C)$ if $x \in Sx$. We denote the set of fixed points of a mapping S by $F(S)$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x.

Given a multi-valued mapping $S : C \longrightarrow CB(C)$, then $(I - S)$ is said to be *demiclosed* at zero if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ imply $x \in Sx$, where I is the identity mapping on C . It is well known that if $S : C \longrightarrow K(C)$ is a multi-valued nonexpansive mapping, then $(I - S)$ is demiclosed at zero, where C is a closed and convex subset of a Hilbert space H. For more details as regards the demiclosedness principle for nonexpansive mappings (see [1]).

Definition 1.2 *Let $S : C \longrightarrow CB(C)$ be a multi-valued mapping. Then, S is said to be*

1. *quasi-nonexpansive if $F(S) \neq \emptyset$ and for all $p \in F(S)$, $x \in C$, we have*

$$D(Sx, Sp) \leq \|x - p\|.$$

2. *demicontractive-type if $F(S) \neq \emptyset$, there exists a constant $k \in [0, 1)$ and for all $p \in F(S)$, $x \in C$, we have*

$$D^2(Sx, Sp) \leq \|x - p\|^2 + \|x - u\|^2,\ \forall\, u \in Sx.$$

3. *hemicontractive-type if $F(S) \neq \emptyset$ and for all $p \in F(S)$, $x \in C$,*

$$D^2(Sx, Sp) \leq \|x - p\|^2 + \|x - u\|^2,\ \forall\, u \in Sx.$$

We observe that every nonexpansive mapping with nonempty set of fixed points is quasi-nonexpansive mapping, every $k-$strictly pseudocontractive mapping S with $F(S) \neq \emptyset$ and $S(p) = \{p\},\ \forall p \in F(S)$ is demicontractive-type mapping, and every pseudocontractive mapping S with $F(S) \neq \emptyset$ and $S(p) = \{p\},\ \forall p \in F(S)$ is hemicontractive-type mapping. It is also easy to see that every quasi-nonexpansive mapping is demicontractive-type mapping, and every demicontractive-type mapping is hemicontractive-type mapping. However, the inclusions are proper (see, *for example*, [15, 27]).

Many authors have shown their interest in studying the existence and approximation of fixed points of non-linear mappings (including hemicontractive-type mapping) (see, *for example*, [6, 12, 20, 21, 27] and the references therein).

Recall that a mapping $A : C \longrightarrow H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2,\ \forall\, x, y \in C.$$

The mapping A is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall\, x, y \in C.$$

Clearly, the class of monotone mappings includes the class of α-inverse strongly monotone mappings but the inclusion is proper (see [15]).

Note that every α-inverse strongly monotone mapping A is $\frac{1}{\alpha}-$Lipschitz mapping and for each $\lambda \in (0, 2\alpha]$, $I - \lambda A$ is a nonexpansive mapping from C into H (see, *for example*, [15, 25]). It is known that if $S : C \longrightarrow H$ is nonexpansive, then $A := I - S$ is $\frac{1}{2}-$inverse strongly monotone mapping (*for more details*, see [23]).

Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \longrightarrow H$ be a nonlinear mapping. The *classical variational inequality problem*, which was developed as a tool for solving partial differential equations by Stampacchia [22], is the problem of finding $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \text{ for all } v \in C. \tag{1.2}$$

The set of solutions of the variational inequality problem (1.2) is denoted by $VI(C, A)$. Variational inequality problem encompasses many mathematical problems, among others, including nonlinear equations, complementarity problems and fixed point problems. Variational inequality problems have been extensively studied by several authors (see, *for example*, [2, 10, 14, 26, 31, 32] and the references cited therein).

Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem*, which was initially introduced by Blum and Oettli [3] in 1994, is the problem of finding a point $x \in C$ such that

$$F(x, y) \geq 0, \text{ for all } y \in C. \tag{1.3}$$

The set of solutions of problem (1.3) is denoted by $EP(F)$. Various problems arising in physics, optimization, economics, engineering and transportation can be reduced to finding solutions of equilibrium problem. Many authors have considered an iterative algorithm for approximating solutions of equilibrium problems (1.3) (see, *for example*, [7, 15, 16, 24, 26] and the references cited therein). The equilibrium problem (1.3) includes variational inequality problems, optimization problems, Nash equilibrium problems and fixed point problems as special cases. As a generalization of the equilibrium problem (1.3), He [9] introduced the following class of split equilibrium problems that enable us to find a solution of one equilibrium problem such that its image under a given bounded linear operator is a solution of another equilibrium problem in different subsets of spaces.

Throughout the rest of this paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces and let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively.

Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions and let $B : H_1 \longrightarrow H_2$ be a bounded linear operator, then the *split equilibrium problem (SEP)* is the problem of finding a point $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \text{ for all } x \in C\,, \tag{1.4}$$

and such that

$$y^* = Bx^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \text{ for all } y \in Q\,. \tag{1.5}$$

In this work we denote the set of solutions of split equilibrium problem (1.4) and (1.5) by Ω. That is,

$$\Omega = \{p \in C : p \in EP(F_1) \text{ and } Bp \in EP(F_2)\}.$$

Given two nonlinear mappings $A_1 : C \longrightarrow H_1$ and $A_2 : Q \longrightarrow H_2$. If we take $F_1(x,y) = \langle A_1 x, y - x\rangle$ for all $x, y \in C$ and $F_2(u,v) = \langle A_2 u, v - u\rangle$ for all $u, v \in Q$, then the split equilibrium problem (SEP) condense to split variational inequality problem (SVIP), that is, to find a point $x^* \in C$ such that

$$x^* \in VI(C, A_1) \text{ and such that } Bx^* \in VI(Q, A_2),$$

which was studied by Censor et al.[4].

Let $S : C \longrightarrow C$ and $T : Q \longrightarrow Q$ be given nonlinear mappings. If we define $F_1(x,y) = \langle (I - S)x, y - x\rangle$ for all $x, y \in C$ and $F_2(u,v) = \langle (I - T)u, v - u\rangle$ for all $u, v \in Q$, then the split equilibrium problem (SEP) reduces to split fixed point problem (SFPP), which is formulated as finding a point $x^* \in C$ with the property:

$$x^* \in F(S) \text{ such that } Bx^* \in F(T),$$

which was studied by Censor and Segal [5], and Eslamian [8].

On the other hand, if we let $H_1 = H_2$, $B = I$, $Q = C$ and $F_2 \equiv 0$, then the split equilibrium problem reduces to the equilibrium problem (1.3). Hence, split equilibrium problem includes split variational inequality, split fixed point problems and the classical equilibrium problems as a special case. Thus, the split equilibrium problem is quite general.

For finding a solution of split equilibrium problems, He [9] also considered the following iterative algorithm:

$$\begin{cases} x_1 \in C, \\ F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ y \in C, \\ F_2(w_n, z) + \frac{1}{r_n}\langle z - w_n, w_n - Bu_n\rangle \geq 0, \ z \in Q, \\ x_{n+1} = P_C(u_n + \mu B^*(w_n - Bu_n)), \ \forall n \geq 1, \end{cases} \tag{1.6}$$

where B^* is the adjoint of the bounded linear operator B. Then, under mild conditions on parameters $\{r_n\}$ and μ, the author proved that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.6) converge weakly to a point $p \in \Omega$.

Motivated by the works of He [9], Kazmi and Rizvi [11] studied the problem of finding a common point of the set of fixed points of nonexpansive single-valued self-mapping $S : C \longrightarrow C$, the sets of solutions of split equilibrium and variational inequality problems by considering the following iterative algorithm:

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma B^*(T_{r_n}^{F_2} - I)Bx_n, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n, \ \forall n \geq 0, \end{cases} \tag{1.7}$$

where $A : C \longrightarrow H_1$ is an $\alpha-$inverse strongly monotone mapping, F_2 is upper semi-continuous in the first argument and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy some fitting conditions. Then, they proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $p = P_\Theta v$, where $\Theta = F(S) \cap \Omega \cap VI(C, A)$.

We remark that the result of Kazmi and Rizvi [11] improves and extends the result of He [9], however, it is restricted to single-valued nonexpansive and $\alpha-$inverse monotone mappings; and the assumption F_2 is upper semi-continuous in the first argument is strong.

However, recently, Meche et al.[17] observed that the condition F_2 is upper semi-continuous assumed in [11] can be dispensed with, and they investigated the following iterative algorithm for approximating a common solution of split equilibrium problem, variational inequality problem for Lipschitz monotone mapping $A : C \longrightarrow H_1$ and fixed point problem for nonexpansive multi-valued mapping S:

$$\begin{cases} x_0 \in C, \\ z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ u_n = P_C[z_n - \gamma_n A z_n], \\ y_n = P_C[z_n - \gamma_n A u_n], \\ x_{n+1} = a_n f(x_n) + (1-a_n)(b_n x_n + (1-b_n)v_n), \ \forall n \geq 0, \end{cases} \tag{1.8}$$

where $B : H_1 \longrightarrow H_2$ is a bounded linear operator and B^* is the adjoint of B, $v_n \in Sy_n$, $f : C \longrightarrow C$ is a contraction mapping , and $\{\gamma_n\} \subset (0, \frac{1}{L})$, $\{b_n\}, \{a_n\} \subset (0,1)$ satisfying some appropriate restrictions. They proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to a point $p \in \Theta = F(S) \cap \Omega \cap VI(C, A)$, where $p = P_\Theta f(p)$.

Motivated and inspired by the above results, we have raised the following natural question:

Question: *Is it possible to introduce an iterative algorithm which converges strongly to a common element of the set of fixed points of class of multi-valued mappings more general than nonexpansive mappings, the set of solutions of variational inequality problem and the set of solutions of split equilibrium problem*?

It is our purpose in this paper to construct an iterative algorithm and prove that the algorithm converges strongly to a common element of fixed point set of a Lipschitz hemicontractive-type multi-valued mapping, solution sets of split equilibrium and variational inequality problems in the framework of real Hilbert spaces. The results presented in this paper extend and improve the corresponding results announced by Censor et al.[4], Eslamian [8], He [9], Kazmi and Rizvi [11], Meche et al.[14, 17] and some other results in this research field.

2 Preliminaries

In this section, we collect some concepts and results that play a crucial role in the sequel.

Let $S : C \longrightarrow C$ be nonexpansive mapping with $F(S) \neq \emptyset$. Then, for every $x \in C$ and $y \in F(S)$, we obtain that

$$\langle x - Sx, y - Sx \rangle \leq \frac{1}{2}\|Sx - x\|^2, \tag{2.1}$$

(see e.g., [17]).

Recall that for every point x in a real Hilbert space H, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

The mapping $P_C : H \longrightarrow 2^C$ is called the metric projection of H onto C and it is characterized by

$$z = P_C x \in C \text{ if and only if } \langle x - z, z - y \rangle \geq 0, \text{ for all } x \in H, y \in C, \tag{2.2}$$

In what follows, we shall make use of the following assumption. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

$(A1)$ $F(x,x) = 0, \forall\, x \in C$;

$(A2)$ F is monotone, i.e., $F(x,y) + F(y,x) \leq 0, \forall\, x,y \in C$;

$(A3)$ $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y), \forall\, x,y,z \in C$;

$(A4)$ for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

In the proof of our main result, we make use the following lemmas.

Lemma 2.1 *[28] Let $\{b_n\}$ be a sequence of nonnegative real numbers such that*

$$b_{n+1} \leq (1-\alpha_n)b_n + \alpha_n\delta_n, \textit{ for } n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\delta_n \subset \mathbb{R}$ satisfying the following restrictions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \textit{ and } \limsup_{n\to\infty} \delta_n \leq 0.$$

Then, $\lim_{n\to\infty} b_n = 0$.

Lemma 2.2 *[30] Let H be a real Hilbert space. Then, for all $x_i \in H$ and $\alpha_i \in [0,1]$, for $i = 1,2,3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ the following equality holds:*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\|^2 = \sum_{i=1}^{3} \alpha_i \|x_i\|^2 - \sum_{1\leq i,j\leq 3} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3 *Let H be a real Hilbert space. Then, for every $x,y \in H$, we have the following:*

$$\begin{aligned} &i) \quad \|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle. \\ &ii) \quad \|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \end{aligned}$$

Lemma 2.4 *[18] Let H be a Hilbert space. Let $A, B \in CB(H)$ and $a \in A$. Then, for $\varepsilon > 0$, there exists a point $b \in B$ such that $||a-b|| \leq D(A,B) + \varepsilon$. In particular, for every $a \in A$ there exists an element $b \in B$ such that $||a-b|| \leq 2D(A,B)$.*

Lemma 2.5 *[3, 7] Let F_1 be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying Assumption 2. For $s > 0$ and for all $x \in H_1$, define a mapping $T_s^{F_1} : H_1 \longrightarrow C$ as follows:*

$$T_s^{F_1} x = \{z \in C : F_1(z,y) + \frac{1}{s}\langle y - z, z - x\rangle \geq 0, \ \forall\, y \in C\}.$$

Then, we have the following:

1. *$T_s^{F_1}$ is nonempty and single valued;*
2. *$T_s^{F_1}$ is firmly nonexpansive, i.e., $\|T_s^{F_1}x - T_s^{F_1}y\|^2 \leq \langle T_s^{F_1}x - T_s^{F_1}y, x-y\rangle, \forall\, x,y \in H_1$;*
3. *$F(T_s^{F_1}) = EP(F_1)$;*
4. *$EP(F_1)$ is closed and convex.*

Furthermore, assume that $F_2 : Q \times Q \longrightarrow \mathbb{R}$ satisfies Assumption 2. For $r > 0$ and for all $u \in H_2$, define a mapping $T_r^{F_2} : H_2 \longrightarrow Q$ as follows:

$$T_r^{F_2} u = \{w \in Q : F_2(w,v) + \frac{1}{r}\langle v-w, w-u\rangle \geq 0,\ \forall\, v \in Q\}.$$

Then, the following hold:

i. $T_r^{F_2}$ is nonempty and single valued;

ii. $T_r^{F_2}$ is firmly nonexpansive, i.e., $\|T_r^{F_2}u - T_r^{F_2}v\|^2 \leq \langle T_r^{F_1}u - T_r^{F_2}v, u-v\rangle$, $\forall\, u,v \in H_2$;

iii. $F(T_r^{F_2}) = EP(F_2)$;

iv. $EP(F_2)$ is closed and convex.

Lemma 2.6 *[29] Let A be a continuous monotone mapping from C into H_1. Then, for any $t > 0$ and $x \in H_1$, there exists $z \in C$ such that*

$$\langle Az, y-z\rangle + \frac{1}{t}\langle y-z, z-x\rangle \geq 0,\ \forall\, y \in C.$$

Moreover, the mapping $J_t : H_1 \longrightarrow C$ defined by

$$J_t x = \{z \in C : \langle Az, y-z\rangle + \frac{1}{t}\langle y-z, z-x\rangle \geq 0,\ \forall\, y \in C\}$$

satisfies the following:

1. *J_t is single-valued;*

2. *J_t is firmly nonexpansive, i.e., $\|J_t x - J_t y\|^2 \leq \langle J_t x - J_t y, x-y\rangle$, $\forall\, x,y \in H_1$;*

3. *$F(J_t) = VI(C,A)$;*

4. *$VI(C,A)$ is closed and convex.*

Lemma 2.7 *[13] Let $\{a_n\}$ be a sequence of real numbers such that there exist a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

3 Main Results

In this section, we prove strong convergence theorems for a split equilibrium problem, variational inequality problem and a fixed point problem for a Lipschitz hemicontractive-type multi-valued mapping.

Theorem 3.1 *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : C \longrightarrow H_1$ be a continuous monotone mapping and $B : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz hemicontractive-type multi-valued mapping. Assume that $\Theta = F(S) \cap \Omega \cap VI(C, A)$ is nonempty and $Sp = \{p\}$ for all $p \in \Theta$. Let $x_0, u \in C$ be arbitrary and let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ y_n = J_t z_n, \\ u_n = (1 - \alpha_n)y_n + \alpha_n v_n, \\ x_{n+1} = a_n u + b_n w_n + c_n y_n, \end{cases} \tag{3.1}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$, $s, r, t > 0$, $\lambda \in (0, \frac{1}{d})$, $d = \|B^*B\|$, where B^* is the adjoint of B, $\{a_n\}, \{\alpha_n\} \subset (0, 1)$, and $\{b_n\},\{c_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ satisfying the following control conditions:
i) $a_n + b_n + c_n = 1$; ii) $a_n + b_n \leq \alpha_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ is bounded.

First, we show that $B^*(I - T_r^{F_2})B$ is a $\frac{1}{2d}-$inverse strongly monotone mapping. Since $T_r^{F_2}$ is nonexpansive, we have $I - T_r^{F_2}$ is $\frac{1}{2}-$inverse strongly monotone mapping. Thus, for all $x, y \in H_1$, from Cauchy Schwartz inequality, we get that
$\|B^*(I - T_r^{F_2})Bx - B^*(I - T_r^{F_2})By\|^2$

$$\begin{aligned} &= \langle B^*((I - T_r^{F_2})Bx - (I - T_r^{F_2})By), B^*((I - T_r^{F_2})Bx - (I - T_r^{F_2})By)\rangle \\ &= \langle (I - T_r^{F_2})Bx - (I - T_r^{F_2})By, BB^*((I - T_r^{F_2})Bx - (I - T_r^{F_2})By)\rangle \\ &\leq d\|(I - T_r^{F_2})Bx - (I - T_r^{F_2})By\|^2 \\ &\leq 2d\langle Bx - By, (I - T_r^{F_2})Bx - (I - T_r^{F_2})By\rangle \\ &= 2d\langle x - y, B^*(I - T_r^{F_2})Bx - B^*(I - T_r^{F_2})By\rangle, \end{aligned}$$

which implies that $B^*(I - T_r^{F_2})B$ is a $\frac{1}{2d}-$inverse strongly monotone mapping. Since $\lambda \in (0, \frac{1}{d})$, we get $I - \lambda B^*(I - T_r^{F_2})B$ is nonexpansive. Hence, from the nonexpansiveness of $T_s^{F_1}$, we obtain that

$$\|T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x - T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)y\| \leq \|x - y\|. \tag{3.2}$$

Now, Let $p \in \Theta$. Then, we have $Sp = p$, $J_t p = p$ and $p \in \Omega$, hence $p = T_s^{F_1}p$ and $Bp = T_r^{F_2}Bp$ which implies that $T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)p = p$. Thus, using (3.2), we get that

$$\|z_n - p\| = \|T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n - p\| \leq \|x_n - p\|. \tag{3.3}$$

Since J_t is nonexpansive, from (3.3) we have that

$$\begin{aligned} \|y_n - p\| &= \|J_t z_n - J_t p\| \leq \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.4}$$

Since S is hemicontractive-type mapping and $w_n \in Su_n$, we get that

$$\begin{aligned} \|w_n - p\|^2 &\leq D^2(Su_n, Tp) \\ &\leq \|u_n - p\|^2 + \|u_n - w_n\|^2. \end{aligned} \tag{3.5}$$

On the other hand, Since S is hemicontractive-type mapping and $v_n \in Sy_n$, from (3.1), (3.4) and Lemma 2.2, we obtain that

$$\begin{aligned} \|u_n - p\|^2 &= \|(1-\alpha_n)y_n + \alpha_n v_n - p\|^2 \\ &= (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\|v_n - p\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n D^2(Sy_n, Sp) \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\Big(\|y_n - p\|^2 + \|y_n - v_n\|^2\Big) \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &= \|y_n - p\|^2 + \alpha_n\|y_n - v_n\|^2 - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &\leq \|x_n - p\|^2 + \alpha_n^2\|y_n - v_n\|^2. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we obtain that

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 + \alpha_n^2\|y_n - v_n\|^2 + \|u_n - w_n\|^2. \tag{3.7}$$

From the assumption that $\|v_n - w_n\| \leq 2D(Ty_n, Tu_n)$ and S is $L-$Lipschitzian mapping, and Lemma 2.2, we find that

$$\begin{aligned} \|u_n - w_n\|^2 &= \|(1-\alpha_n)(y_n - w_n) + \alpha_n(v_n - w_n)\|^2 \\ &= (1-\alpha_n)\|y_n - w_n\|^2 + \alpha_n\|v_n - w_n\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &\leq (1-\alpha_n)\|y_n - w_n\|^2 + 4\alpha_n D^2(Ty_n, Tu_n) \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &\leq (1-\alpha_n)\|y_n - w_n\|^2 + 4\alpha_n L^2\|y_n - u_n\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &= (1-\alpha_n)\|y_n - w_n\|^2 + 4\alpha_n^3 L^2\|y_n - v_n\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\ &= (1-\alpha_n)\|y_n - w_n\|^2 + \alpha_n(4L^2\alpha_n^2 + \alpha_n - 1)\|y_n - v_n\|^2. \end{aligned} \tag{3.8}$$

Thus, by substituting (3.8) into (3.7), we obtain that

$$\begin{aligned} \|w_n - p\|^2 &\leq \|x_n - p\|^2 + \alpha_n^2 \|y_n - v_n\|^2 + (1-\alpha_n)\|y_n - w_n\|^2 \\ &\quad + \alpha_n(4L^2\alpha_n^2 + \alpha_n - 1)\|y_n - v_n\|^2 \\ &= \|x_n - p\|^2 + (1-\alpha_n)\|y_n - w_n\|^2 \\ &\quad + \alpha_n(4L^2\alpha_n^2 + 2\alpha_n - 1)\|y_n - v_n\|^2. \end{aligned} \tag{3.9}$$

Hence, from Lemma 2.2, (3.4), (3.9) and condition (i), we have that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|a_n u + b_n w_n + c_n y_n - p\|^2 \\ &\leq a_n\|u - p\|^2 + b_n\|w_n - p\|^2 + c_n\|y_n - p\|^2 - b_n c_n\|y_n - w_n\|^2 \\ &\leq a_n\|u - p\|^2 + b_n\Big(\|x_n - p\|^2 + (1-\alpha_n)\|y_n - w_n\|^2 \\ &\quad + \alpha_n(4L^2\alpha_n^2 + 2\alpha_n - 1)\|y_n - v_n\|^2\Big) \\ &\quad + c_n\|x_n - p\|^2 - b_n c_n\|y_n - w_n\|^2 \\ &= a_n\|u - p\|^2 + (1-a_n)\|x_n - p\|^2 + b_n(1 - c_n - \alpha_n) \\ &\quad \times \|y_n - w_n\|^2 - b_n\alpha_n(1 - 4L^2\alpha_n^2 - 2\alpha_n)\|y_n - v_n\|^2. \\ &\leq a_n\|u - p\|^2 + (1-a_n)\|x_n - p\|^2 \\ &\quad - b_n\alpha_n(1 - 4L^2\alpha_n^2 - 2\alpha_n)\|y_n - v_n\|^2 \\ &\quad + b_n(a_n + b_n - \alpha_n)\|y_n - w_n\|^2, \end{aligned} \tag{3.10}$$

and from condition (ii), we see that

$$1 - 4L^2\alpha_n^2 - 2\alpha_n \geq 1 - 4L^2c^2 - 2c > 0 \text{ and } a_n + b_n - \alpha_n \leq 0, \tag{3.11}$$

for all $n \geq 0$. Thus, combining (3.10) and (3.11), we obtain that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq a_n\|u - p\|^2 + (1-a_n)\|x_n - p\|^2 \\ &\leq \max\{\|u - p\|^2, \|x_n - p\|^2\}. \end{aligned}$$

Then, by induction, we have that

$$\|x_n - p\|^2 \leq \max\{\|u - p\|^2, \|x_0 - p\|^2\}.$$

Therefore, the sequence $\{x_n\}$ is bounded. This completes the proof.

Theorem 3.2 *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : C \longrightarrow H_1$ be a continuous monotone mapping and $B : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz hemicontractive-type multi-valued mapping. Assume that $\Theta = F(S) \cap \Omega \cap VI(C, A)$ is nonempty, closed and convex, $Sp = \{p\}$ for all $p \in \Theta$ and $(I - S)$ is demiclosed at zero. Let $x_0, u \in C$ be arbitrary and let $\{x_n\}$ be a sequence in C*

generated by

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ y_n = J_t z_n, \\ u_n = (1-\alpha_n)y_n + \alpha_n v_n, \\ x_{n+1} = a_n u + b_n w_n + c_n y_n, \end{cases} \tag{3.12}$$

for all $n \geq 0$*, where* $v_n \in Sy_n$ *and* $w_n \in Su_n$ *are such that* $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$*,* $s, r, t > 0$*,* $\lambda \in (0, \frac{1}{d})$*,* $d = \|B^*B\|$*, where* B^* *is the adjoint of* B*,* $\{a_n\}, \{\alpha_n\} \subset (0,1)$*, and* $\{b_n\}, \{c_n\} \subset [a,b]$ *for some* $a, b \in (0,1)$ *satisfying the following control conditions:*
i) $a_n + b_n + c_n = 1$*; ii)* $\lim_{n\to\infty} a_n = 0$*,* $\sum_{n=0}^{\infty} a_n = \infty$*; iii)* $a_n + b_n \leq \alpha_n \leq c < \dfrac{1}{\sqrt{1+4L^2}+1}$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $q \in \Theta$*, where* $q = P_\Theta(u)$*.*

First, we note that P_Θ is well defined because Θ is nonempty, closed and convex subset of C and from Theorem 3.1 it follows that the sequence $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$. Now, let $p \in \Theta$. Then, since $T_s^{F_1}$ is nonexpansive, we have that

$$\begin{aligned} \|z_n - p\|^2 &= \|T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n - T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)p\|^2 \\ &\leq \|(I - \lambda B^*(I - T_r^{F_2})B)x_n - (I - \lambda B^*(I - T_r^{F_2})B)p\|^2 \\ &= \|(x_n - p) - \lambda(B^*(I - T_r^{F_2})Bx_n - B^*(I - T_r^{F_2})Bp)\|^2 \\ &= \|x_n - p\|^2 - 2\lambda\langle x_n - p, B^*(I - T_r^{F_2})Bx_n - B^*(I - T_r^{F_2})Bp\rangle \\ &\quad + \lambda^2\|B^*(I - T_r^{F_2})Bx_n - B^*(I - T_r^{F_2})Bp\|^2. \end{aligned}$$

Hence, the $\frac{1}{2d}-$inverse strong monotonicity of $B^*(I - T_r^{F_2})B$ and the fact that $Bp = T_r^{F_2}Bp$ give that

$$\begin{aligned} \|z_n - p\|^2 &\leq \|x_n - p\|^2 - \frac{\lambda}{d}\|B^*(I - T_r^{F_2})Bx_n - B^*(I - T_r^{F_2})Bp\|^2 \\ &\quad + \lambda^2\|B^*(I - T_r^{F_2})Bx_n - B^*(I - T_r^{F_2})Bp\|^2 \\ &= \|x_n - p\|^2 + \lambda(\lambda - \frac{1}{d})\|B^*(I - T_r^{F_2})Bx_n\|^2. \end{aligned} \tag{3.13}$$

On the other hand, from (3.12), (3.4), Lemma 2.2 and Lemma 2.3 (ii), we get

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|a_n u + b_n w_n + c_n y_n - p\|^2 \\ &\leq \|b_n(w_n - p) + c_n(y_n - p)\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle \\ &\leq b_n\|w_n - p\|^2 + c_n\|y_n - p\|^2 - b_n c_n\|y_n - w_n\|^2 \\ &\quad + 2a_n\langle u - p, x_{n+1} - p\rangle. \\ &\leq b_n\|w_n - p\|^2 + c_n\|z_n - p\|^2 - b_n c_n\|y_n - w_n\|^2 \\ &\quad + 2a_n\langle u - p, x_{n+1} - p\rangle. \end{aligned} \tag{3.14}$$

Therefore, using (3.9), (3.13) and (3.14), we get the following:

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - a_n)\|x_n - p\|^2 - b_n\alpha_n(1 - 4L^2\alpha_n^2 - 2\alpha_n) \\ &\quad \times \|y_n - v_n\|^2 + b_n(a_n + b_n - \alpha_n)\|y_n - w_n\|^2 \\ &\quad - c_n\lambda(\frac{1}{d} - \lambda)\|B^*(I - T_r^{F_2})Bx_n\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle. \end{aligned} \tag{3.15}$$

Then, we complete the proof by the next two cases.

Case 1: Suppose that there exists a positive integer n_0 such that $\{\|x_n - p\|\}$ is decreasing for all $n \geq n_0$. Then, the sequence $\{\|x_n - p\|\}$ is convergent and from (3.11) and (3.15), we have that

$$\begin{aligned} c_n\lambda(\frac{1}{d} - \lambda)\|B^*(I - T_r^{F_2})Bx_n\|^2 &\leq (1-a_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2a_n\langle u - p, x_{n+1} - p\rangle. \end{aligned}$$

Hence, assumption of $\{c_n\}$, convergence of $\{||x_n - p||\}$ and the fact that $a_n \to 0$ as $n \to \infty$ imply that

$$\lim_{n\to\infty} \|B^*(I - T_r^{F_2})Bx_n\| = 0, \tag{3.16}$$

and hence

$$\lim_{n\to\infty} \|x_n - (x_n - \lambda B^*(I - T_r^{F_2})Bx_n)\| = 0. \tag{3.17}$$

And Since $T_s^{F_1}$ is firmly nonexpansive and $(I - \lambda B^*(I - T_r^{F_2})B)$ is nonexpansive, from (3.1) and Lemma 2.3 (i), we have that

$$\begin{aligned} \|z_n - p\|^2 &= \|T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n - T_s^{F_1}p\|^2 \\ &\leq \langle z_n - p, (I - \lambda B^*(I - T_r^{F_2})B)x_n - p\rangle \\ &= \frac{1}{2}\Big(\|z_n - p\|^2 + \|(I - \lambda B^*(I - T_r^{F_2})B)x_n - p\|^2 \\ &\quad - \|z_n - (I - \lambda B^*(I - T_r^{F_2})B)x_n\|^2\Big) \\ &\leq \frac{1}{2}\Big(\|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2 \\ &\quad - 2\lambda\langle z_n - x_n, B^*(I - T_r^{F_2})Bx_n\rangle - \lambda^2\|B^*(I - T_r^{F_2})Bx_n\|^2\Big). \end{aligned}$$

But this

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - x_n\|^2 + 2\lambda\langle x_n - z_n, B^*(I - T_r^{F_2})Bx_n\rangle. \tag{3.18}$$

Thus, substituting (3.9) and (3.18) into (3.14), we obtain that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1-a_n)\|x_n - p\|^2 - b_n\alpha_n(1 - 4L^2\alpha_n^2 - 2\alpha_n) \times \|y_n - v_n\|^2 \\ &\quad + b_n(a_n + b_n - \alpha_n)\|y_n - w_n\|^2 - c_n\|z_n - x_n\|^2 \\ &\quad + 2\lambda c_n\langle x_n - z_n, B^*(I - T_r^{F_2})Bx_n\rangle + 2a_n\langle u - p, x_{n+1} - p\rangle. \end{aligned} \tag{3.19}$$

It follows from (3.11) and (3.19) that

$$\begin{aligned} c_n\|z_n - x_n\|^2 &\leq (1-a_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2c_n\lambda\|x_n - z_n\| \times \|B^*(I - T_r^{F_2})Bx_n\| + 2a_n\langle u - p, x_{n+1} - p\rangle. \end{aligned}$$

Hence, since $\{x_n\}$ and $\{z_n\}$ are bounded and $a_n \to 0$ as $n \to \infty$, from (3.16) and assumption of $\{c_n\}$, we obtain that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.20}$$

On the other hand, since J_t is firmly nonexpansive, from Lemma 2.3 (i), we get that

$$\begin{aligned}\|y_n - p\|^2 &= \|J_t z_n - J_t p\|^2\\ &\leq \langle y_n - p, z_n - p\rangle\\ &= \frac{1}{2}\left(\|y_n - p\|^2 + \|z_n - p\|^2 - \|z_n - y_n\|^2\right).\end{aligned}$$

This together with (3.3) give that

$$\begin{aligned}\|y_n - p\|^2 &\leq \|z_n - p\|^2 - \|z_n - y_n\|^2\\ &\leq \|x_n - p\|^2 - \|z_n - y_n\|^2. \end{aligned}\tag{3.21}$$

Then, by substituting (3.9) and (3.21) into (3.14), we obtain that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - a_n)\|x_n - p\|^2 - b_n\alpha_n(1 - 4L^2\alpha_n^2 - 2\alpha_n)\\ &\quad \times \|y_n - v_n\|^2 + b_n(a_n + b_n - \alpha_n)\|y_n - w_n\|^2\\ &\quad - c_n\|z_n - y_n\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle. \end{aligned}\tag{3.22}$$

Thus, the fact that $a_n + b_n - \alpha_n \leq 0$ and (3.22) yield that

$$\begin{aligned}b_n\alpha_n(1 - 4L^2\alpha_n^2 - 2\alpha_n)\|y_n - v_n\|^2 &\leq (1 - a_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\quad + 2a_n\langle u - p, x_{n+1} - p\rangle.\end{aligned}$$

Hence, the assumption that $\alpha_n \to 0$ as $n \to \infty$ and (3.11) imply that

$$\lim_{n\to\infty} \|y_n - v_n\| = 0, \tag{3.23}$$

and hence since $v_n \in Sy_n$, from (3.23), we infer that

$$\lim_{n\to\infty} d(y_n, Sy_n) = 0. \tag{3.24}$$

In addition, from (3.11) and (3.22), we also have

$$\begin{aligned}c_n\|y_n - z_n\|^2 &\leq (1 - a_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\quad + 2a_n\langle u - p, x_{n+1} - p\rangle.\end{aligned}$$

Because $\alpha_n \to 0$ as $n \to \infty$, it follows from the assumption of $\{c_n\}$ that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.25}$$

And from the Lipschitz condition of S, (3.12) and (3.23), we get that

$$\begin{aligned}\|y_n - w_n\| &\leq \|y_n - v_n\| + \|v_n - w_n\|\\ &\leq \|y_n - v_n\| + 2L\|y_n - u_n\|\\ &= \|y_n - v_n\| + 2L\alpha_n\|y_n - v_n\| \to 0 \text{ as } n \to \infty. \end{aligned}\tag{3.26}$$

Therefore, from (3.20), (3.25), (3.26), definition of $\{x_{n+1}\}$ and the assumption that $\alpha_n \to 0$ as $n \to \infty$, we find that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\ &= \|a_n(u - y_n) + b_n(w_n - y_n)\| + \|y_n - z_n\| + \|z_n - x_n\| \\ &\leq a_n\|u - y_n\| + b_n\|w_n - y_n\| + \|y_n - z_n\| + \|z_n - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.27}$$

Moreover, from (3.11) and (3.22), we infer that

$$\|x_{n+1} - p\|^2 \leq (1 - a_n)\|x_n - p\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle. \tag{3.28}$$

Now, let $q = P_\Theta(u)$. Then, we show that $\limsup_{n\to\infty}\langle u - q, x_{n+1} - q\rangle \leq 0$.
Since the sequence $\{x_{n+1}\}$ is bounded in a real Hilbert space H_1, we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \rightharpoonup w$ as $i \to \infty$ and

$$\limsup_{n\to\infty}\langle u - q, x_{n+1} - q\rangle = \lim_{i\to\infty}\langle u - q, x_{n_i+1} - q\rangle.$$

Since C is closed and convex, C is weakly closed. So, we have $w \in C$ and from (3.27), we find that $x_{n_i} \rightharpoonup w$ as $i \to \infty$ and thus it follows from (3.20) and (3.25) that

$$z_{n_i} \rightharpoonup w \text{ and } y_{n_i} \rightharpoonup w \text{ as } i \to \infty.$$

Next, we claim that $w \in \Theta$. From (3.24) and the hypothesis that $(I - S)$ is demiclosed at zero, we obtain that

$$w \in F(S).$$

Since $(I - \lambda B^*(I - T_r^{F_2})B)$ is nonexpansive, from (3.17) and demiclosedness princeple for nonexpansive, we get that

$$w = (I - \lambda B^*(I - T_r^{F_2})B)w,$$

which implies that

$$B^*(I - T_r^{F_2})Bw = 0.$$

Thus, using (2.1) we obtain that $Bw = T_r^{F_2}Bw$, hence $Bw \in EP(F_2)$. In addition, from (3.20), we have

$$\lim_{i\to\infty}\|x_{n_i} - T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_{n_i}\| = \lim_{i\to\infty}\|x_{n_i} - z_{n_i}\| = 0.$$

Hence, since $T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)$ is nonexpansive, from the demiclosedness of a nonexpansive mapping, we obtain that

$$w = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)w.$$

This with the fact that $Bw = T_r^{F_2}Bw$ gives that $w = T_s^{F_1}w$, and hence $w \in EP(F_1)$. Therefore,

$$w \in \Omega.$$

On the other hand, from (3.1) and (3.25), we have

$$\lim_{n\to\infty}\|z_{n_i} - J_t z_{n_i}\| = \lim_{n\to\infty}\|z_{n_i} - y_{n_i}\| = 0.$$

Since $z_{n_i} \rightharpoonup w$ and J_t is nonexpansive, then $(I - J_t)$ is demiclosed at zero and so, we get that

$$w = J_t w \text{ and hence } w \in VI(C, A).$$

Therefore, $w \in \Theta$. Thus, since $q = P_\Theta(u)$ and $x_{n_i+1} \rightharpoonup w$, from the property of metric projection P_C given in (2.2), we have

$$\begin{aligned} \limsup_{n\to\infty} \langle u - q, x_{n+1} - q\rangle &= \lim_{i\to\infty} \langle u - q, x_{n_i+1} - q\rangle \\ &= \langle u - q, w - q\rangle \le 0. \end{aligned} \tag{3.29}$$

Furthermore, since $p \in \Theta$ was arbitrary and $q \in \Theta$, then from (3.28), (3.29) and Lemma 2.1, we get that

$$\|x_n - q\| \to 0 \text{ as } n \to \infty .$$

Consequently, $x_n \to q = P_\Theta(u)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - p\| < \|x_{n_j+1} - p\|,$$

for all $j \in \mathbb{N}$. Then, by Lemma 2.7, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\|x_{m_k} - p\| \le \|x_{m_k+1} - p\| \text{ and } \|x_k - p\| \le \|x_{m_k+1} - p\|, \tag{3.30}$$

for all $k \in \mathbb{N}$. Thus, from (3.11), (3.19), (3.22), (3.30) and the hypothesis that $a_n \to 0$, we get that

$$||z_{m_k} - x_{m_k}|| \to 0, ||y_{m_k} - v_{m_k}|| \to 0 \text{ and } ||y_{m_k} - z_{m_k}|| \to 0 \text{ as } k \to \infty .$$

Then, since $q = P_\Theta(u)$, following the processes in Case 1, we get that

$$\limsup_{k\to\infty} \langle u - q, x_{m_k+1} - q\rangle \le 0. \tag{3.31}$$

Now, since $q \in \Theta$, from (3.22), we have that

$$\|x_{m_k+1} - q\|^2 \le (1 - a_{m_k})\|x_{m_k} - q\|^2 + 2a_{m_k}\langle u - q, x_{m_k+1} - q\rangle, \tag{3.32}$$

and hence (3.30) and (3.32) imply that

$$\begin{aligned} a_{m_k}\|x_{m_k} - q\|^2 &\le \|x_{m_k} - q\|^2 - \|x_{m_k+1} - q\|^2 + 2a_{m_k}\langle u - q, x_{m_k+1} - q\rangle \\ &\le 2a_{m_k}\langle u - q, x_{m_k+1} - q\rangle. \end{aligned}$$

Hence, in view of the fact that $a_{m_k} > 0$, we have that

$$\|x_{m_k} - q\|^2 \le 2\langle u - q, x_{m_k+1} - q\rangle.$$

Thus, using (3.31) we obtain that $\|x_{m_k} - q\| \to 0$ as $k \to \infty$. This together with (3.32) implies that $\|x_{m_k+1} - q\| \to 0$ as $k \to \infty$. Because $\|x_k - q\| \le \|x_{m_k+1} - q\|$ for all $k \in \mathbb{N}$, we get that $x_k \to q$. Therefore, from the above two cases, we deduce that the sequence $\{x_n\}$ converges strongly to $q = P_\Theta(u)$. This completes the proof.

If, in Theorem 3.2, we assume that S is a single-valued Lipschitz hemicontractive mapping, then we obtain the following result:

Corollary 3.3 *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : C \longrightarrow H_1$ be a continuous monotone mapping and $B : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2. Let $S : C \longrightarrow C$ be a Lipschitz hemicontractive mapping with Lipschitz constant L. Assume that $\Theta = F(S) \cap \Omega \cap VI(C, A)$ is nonempty, closed and convex, and $(I - S)$ is demiclosed at zero. Let $x_0, u \in C$ be arbitrary and let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ y_n = J_t z_n, \\ u_n = (1 - \alpha_n)y_n + \alpha_n S y_n, \\ x_{n+1} = a_n u + b_n S u_n + c_n y_n, \end{cases}$$

for all $n \geq 0$, where $s, r, t > 0$, $\lambda \in (0, \frac{1}{d})$, $d = \|B^*B\|$, where B^* is the adjoint of B, $\{a_n\}, \{\alpha_n\} \subset (0, 1)$, and $\{b_n\}, \{c_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ satisfying the following control conditions:
$i)$ $a_n + b_n + c_n = 1$; $ii)$ $\lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$; $iii)$ $a_n + b_n \leq \alpha_n \leq c < \frac{1}{\sqrt{1 + L^2} + 1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_\Theta(u)$.

If, in Theorem 3.2, we assume that $A \equiv 0$, then we find the following result on split equilibrium problem and fixed point problem for Lipschitz hemicontractive-type multi-valued mapping.

Corollary 3.4 *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $B : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz hemicontractive-type multi-valued mapping. Assume that $\Theta = F(S) \cap \Omega$ is nonempty, closed and convex, $Sp = \{p\}$ for all $p \in \Theta$ and $(I - S)$ is demiclosed at zero. Let $x_0, u \in C$ be arbitrary and let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ u_n = (1 - \alpha_n)z_n + \alpha_n v_n, \\ x_{n+1} = a_n u + b_n w_n + c_n z_n, \end{cases}$$

for all $n \geq 0$, where $v_n \in Sz_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sz_n, Su_n)$, $s, r > 0$, $\lambda \in (0, \frac{1}{d})$, $d = \|B^*B\|$, where B^* is the adjoint of B, $\{a_n\}, \{\alpha_n\} \subset (0, 1)$, and $\{b_n\}, \{c_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ satisfying the following control conditions:
$i)$ $a_n + b_n + c_n = 1$; $ii)$ $\lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$; $iii)$ $a_n + b_n \leq \alpha_n \leq c < \frac{1}{\sqrt{1 + 4L^2} + 1}$. Then, the sequence $\{x_n\}$ converges strongly to $q = P_\Theta(u)$.

If, in Theorem 3.2, we assume that $H_1 = H_2$, $C = Q$, $B \equiv I$ and $F_2 \equiv 0$, then we obtain the following corollary:

Corollary 3.5 *Let H_1 be a real Hilbert space. Let C be a nonempty, closed and convex subset of H_1. Let $A : C \longrightarrow H_1$ be a continuous monotone mapping and let $F_1 : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 2. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz hemicontractive-type multi-valued mapping. Assume that $\Theta = F(S) \cap EP(F_1) \cap VI(C, A)$ is nonempty, closed and convex, $Sp = \{p\}$ for all $p \in \Theta$ and $(I - S)$ is demiclosed at zero. Let $x_0, u \in C$ be arbitrary and let $\{x_n\}$ be a* *sequence in C generated by*

$$\begin{cases} z_n = T_s^{F_1} x_n, \\ y_n = J_t z_n, \\ u_n = (1-\alpha_n) y_n + \alpha_n v_n, \\ x_{n+1} = a_n u + b_n w_n + c_n y_n, \end{cases}$$

for all $n \geq 0$, *where* $v_n \in Sy_n$ *and* $w_n \in Su_n$ *are such that* $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$, $s, t > 0$, $\{a_n\}, \{\alpha_n\} \subset (0,1)$, *and* $\{b_n\}, \{c_n\} \subset [a,b]$ *for some* $a, b \in (0,1)$ *satisfying the following control conditions:*
i) $a_n + b_n + c_n = 1$; *ii)* $\lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$; *iii)* $a_n + b_n \leq \alpha_n \leq c < \dfrac{1}{\sqrt{1+4L^2}+1}$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$.

If, in Corollary 3.3, we assume that S is an identity mapping, then we get the following result on variational inequality and split equilibrium problems.

Corollary 3.6 *Let* H_1 *and* H_2 *be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of* H_1 *and* H_2, *respectively. Let* $A : C \longrightarrow H_1$ *be a continuous monotone mapping and* $B : H_1 \longrightarrow H_2$ *be a bounded linear operator. Let* $F_1 : C \times C \longrightarrow \mathbb{R}$ *and* $F_2 : Q \times Q \longrightarrow \mathbb{R}$ *be bifunctions satisfying Assumption 2. Assume that* $\Theta = VI(C, A) \cap \Omega$ *is nonempty. Let* $x_0, u \in C$ *be arbitrary and let* $\{x_n\}$ *be a sequence in C generated by*

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B) x_n, \\ y_n = J_t z_n, \\ x_{n+1} = a_n u + (1-a_n) y_n, \end{cases}$$

for all $n \geq 0$, *where* $s, r, t > 0$, $\lambda \in (0, \frac{1}{d})$, $d = \|B^* B\|$, *where* B^* *is the adjoint of B, and* $\{a_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = \infty$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$.

If, in Corollary 3.5, we assume that $F_1 \equiv 0$, then we obtain the following corollary:

Corollary 3.7 *Let* H_1 *be a real Hilbert space. Let C be a nonempty, closed and convex subset of* H_1. *Let* $A : C \longrightarrow H_1$ *be a continuous monotone mapping and let* $S : C \longrightarrow CB(C)$ *be a* $L-$*Lipschitz hemicontractive-type multi-valued mapping. Assume that* $\Theta = F(S) \cap VI(C, A)$ *is nonempty, closed and convex,* $Sp = \{p\}$ *for all* $p \in \Theta$ *and* $(I - S)$ *is demiclosed at zero. Let* $x_0, u \in C$ *be arbitrary and let* $\{x_n\}$ *be a sequence in C generated by*

$$\begin{cases} y_n = J_t x_n, \\ u_n = (1-\alpha_n) y_n + \alpha_n v_n, \\ x_{n+1} = a_n u + b_n w_n + c_n y_n, \end{cases}$$

for all $n \geq 0$, *where* $v_n \in Sy_n$ *and* $w_n \in Su_n$ *are such that* $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$, $t > 0$, $\{a_n\}, \{\alpha_n\} \subset (0,1)$, *and* $\{b_n\}, \{c_n\} \subset [a,b]$ *for some* $a, b \in (0,1)$ *satisfying the following control conditions:*
i) $a_n + b_n + c_n = 1$; *ii)* $\lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$; *iii)* $a_n + b_n \leq \alpha_n \leq c < \dfrac{1}{\sqrt{1+4L^2}+1}$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$.

If, in Corollary 3.6, we assume that $A \equiv 0$, then we obtain the following corollary on split equilibrium problem:

Corollary 3.8 *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $B : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2. Assume that Ω is nonempty. Let $x_0, u \in C$ be arbitrary and let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ x_{n+1} = a_n u + (1 - a_n)z_n, \end{cases}$$

for all $n \geq 0$, where $s, r > 0$, $\lambda \in (0, \frac{1}{d})$, $d = \|B^*B\|$, where B^* is the adjoint of B, and $\{a_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(u)$.

We note that, since every quasi-nonexpansive and demicontractive mappings are hemicontractive, the results obtained in this paper for hemicontractive (single and multi-valued) mapping also hold for quasi-nonexpansive and demicontractive mappings provided that the indicated conditions are satisfied. *Again we remark that, since every pseudocontractive multi-valued mapping S with $F(S) \neq \emptyset$ and $S(p) = \{p\}$, $\forall p \in F(S)$ is a hemicontractive-type multi-valued mapping, the results given in this paper also hold for pseudocontractive multi-valued mapping S with $F(S) \neq \emptyset$ and $S(p) = \{p\}$, $\forall p \in F(S)$. Hence, it is also true for nonexpansive and $k-$strictly pseudocontractive multi-valued mappings provided that the specified assumptions are satisfied because every nonexpansive and $k-$strictly pseudocontractive mappings are pseudocontractive mapping.* Our results extend, improve and unify several recent results in the existing literature (e.g., [4, 8, 9, 11, 14, 17] etc) on approximation of common solution of fixed point problem for nonlinear mappings, classical variational inequality and split equilibrium problems. In particular, Theorem 3.2 extends the results of Kazmi and Rizvi [11] and Meche et al.[17] from the class of nonexpansive mappings to more general class of Lipschitz hemicontractive-type multi-valued mappings; and from the class of Lipschitz monotone mapping to more general class of continuous monotone mappings.

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