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Characterizations of near-rings by (α, β)-fuzzy ideals

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ABSTRACT. Aim of this paper is to investigate the characterizations of the concept of (α, β)-fuzzy ideals of left near-rings, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ by using belongs to relation $(\in)$ and quasi-coincidence with relation (q) between fuzzy points and fuzzy sets. An (α, β)-fuzzy ideal, we can consider twelve different types of such structures because there are three types of α and four types of β. But in this paper, mainly the types $(\in, \in), (\in, \in \vee q), (q, \in \vee q), (\in \vee q, \in \vee q)$ are considered. We also discussed some characterizations of $(\in, \in \vee q_k)$-fuzzy subnear-rings and ideals of near-rings.

1 Introduction

A near-ring satisfies all axioms of an associative ring, except commutativity of addition and one of the two distributive laws. The theory of fuzzy sets was initiated by Zadeh[29] in 1965. A new way of fuzzy subgroup, that is the (α, β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das[7] by using the combined concept of 'belongingness'and 'quasi-coincidence'of fuzzy points and fuzzy sets, which was introduced Pu Ming and Liu Ming[24]. In particular, the concept of $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroup[25]. The concepts of fuzzy subnear-rings and fuzzy ideals in near-rings were first proposed by Abou Said[1]. Davvaz[11, 12] introduced the notion of $(\in, \in \vee q)$-fuzzy subnear-rings(R-subgroup, ideals) of a near-ring and also discussed some of its characterizations. Narayanan and Manikantan[22], introduced the notion of $(\in, \in \vee q)$-fuzzy subnear-rings and $(\in, \in \vee q)$-fuzzy ideals of near-rings. Deena and Coumerassane[14] discussed few concepts of $(\in, \in \vee q_k)$-fuzzy subnear-ring and ideals which is a generalization of $(\in, \in \vee q)$-fuzzy

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subnear-rings and ideals. Recently, Davvaz *et al*. introduced the notion of more generalized form of an (α, β)-fuzzy generalized bi-ideals of regular ordered semigroups in [13]. Asghar Khan *et al*. initiated the notion of $(\in, \in \vee q)$-fuzzy interior ideals of ordered semigroups in [4] and also proposed the idea of generalized fuzzy ideals of ordered semigroups in [3] and ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals in [5]. In [6], discussed the notion of new types of fuzzy bi-ideals in ordered semigroups. In [17, 18, 19], Jianming Zhan *et al*. introduced and characterized some properties of new types of fuzzy ideals(Redefined generalized fuzzy ideals, generalized fuzzy ideals) of near-rings. In [26, 27], Young Bae Jun *et al*. introduced the notion of generalized of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras and he also discussed note on (α, β)-fuzzy ideals of hemirings. Recently, Dudeh *et al*.[15], have considered the concept of (α, β)-fuzzy ideals of hemirings and some of its properties. Muhammed Shabir[20], initiated the concept of (α, β)-fuzzy ideals of regular semigroups and some of its characterizations. Our aim of this paper, is to investigate the characterizations of the concept of (α, β)-fuzzy ideals of left near-rings, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ by using belongs to relation $(\in)$ and quasi-coincidence with relation (q) between fuzzy points and fuzzy sets. An (α, β)-fuzzy ideal, we can consider twelve different types of such structures because there are three types of α and four types of β. But in this paper, we consider mainly on the types $(\in, \in), (\in, \in \vee q), (q, \in \vee q), (\in \vee q, \in \vee q)$ and discussed some characterizations of $(\in, \in \vee q_k)$-fuzzy ideal of R and the relation between these generalized fuzzy ideals.

2 Basic definitions and preliminaries

In this section, some relevant definitions and notations are reproduced based on [11].

A *near-ring* is an algebraic system $(R, +, \cdot)$ consisting of a non empty set R together with two binary operations called $+$ and $\cdot$ such that $(R, +)$ is a group not necessarily abelian and $(R, \cdot)$ is a semigroup connected by the following distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ valid, for all $x, y, z \in R$. We will use the word 'near-ring 'to mean 'left near-ring '. We denote xy instead of $x \cdot y$.

An *ideal* I of a near-ring R is a subset of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $RI \subseteq I$, (iii) $(x + a)y - xy \in I$, for any $a \in I$ and $x, y \in R$.

Note that I is a *left ideal* of R if I satisfies (i) and (ii), and *right ideal* of R if I satisfies (i) and (iii).

A fuzzy subset μ of R we mean any map $\mu : R \to [0, 1]$. A fuzzy subset of the form

$$\mu(y) = \begin{cases} t \in (0, 1], & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

is called a *fuzzy point* with *support* x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy subset μ of the same set R, Pu Ming and Liu Ming[24] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \wedge q, \in \vee q\}$. A fuzzy point x_t is said to *belong to* (resp.*quasi-coincident* with) a fuzzy subset μ, written as $x_t \in \mu$(resp. $x_t q \mu$) if $\mu(x) \geq t$(resp. $\mu(x) + t > 1$). The symbol $x_t \in \vee q \mu$ means that $x_t \in \mu$ or $x_t q \mu$. Similarly, $x_t \in \wedge q \mu$ denotes that $x_t \in \mu$ and $x_t q \mu$. $x_t \overline{\in} \mu$ and $x_t \overline{\in \vee q} \mu$ means that $x_t \in \mu$ and $x_t \in \vee q \mu$ do not hold, respectively.

Let μ be a fuzzy subset of R. The set $S_\mu = \{x \in R | \mu(x) > 0\}$ is called the *support* of μ.

3 (α, β)-Fuzzy Ideal Of Near-ring

In what follows let R be a left near-ring and α, β will denote any one of $\in, q, \in \vee q, \in \wedge q$ unless otherwise specified.

Definition 3.1. A fuzzy subset μ of R is called an (α, β)-fuzzy subnear-ring of R where $\alpha \neq \in \wedge q$ if

(1) $x_t \alpha \mu, y_r \alpha \mu \Rightarrow (x+y)_{\min\{t,r\}} \beta \mu$,

(2) $x_t \alpha \mu \Rightarrow (-x)_t \beta \mu$,

(3) $x_t \alpha \mu, y_r \alpha \mu \Rightarrow (xy)_{\min\{t,r\}} \beta \mu$, for all $x, y \in R$ and $t, r \in (0,1]$.

Definition 3.2. A fuzzy subset μ of R is called an (α, β)-fuzzy ideal of R where $\alpha \neq \in \wedge q$ if

(i) $x_t \alpha \mu, y_r \alpha \mu \Rightarrow (x-y)_{\min\{t,r\}} \beta \mu$,

(ii) $x_t \alpha \mu$ and $y \in R \Rightarrow (y+x-y)_t \beta \mu$,

(iii) $y_t \alpha \mu$ and $x \in R \Rightarrow (xy)_t \beta \mu$,

(iv) $z_t \alpha \mu$ and $x, y \in R \Rightarrow ((x+z)y - xy)_t \beta \mu$, for all $x, y, z \in R$ and $t, r \in (0,1]$.

A fuzzy subset which is an (α, β)-fuzzy left and right ideal is called an (α, β)-fuzzy ideal.

Let μ be a fuzzy subset of R such that $\mu(x) \leq 0.5$ for all $x \in R$. Let $x \in R$ and $t \in (0,1]$ be such that $x_t \in \wedge q \mu$. Then $\mu(x) \geq t$ and $\mu(x) + t > 1$. It follows that $1 < \mu(x) + t \leq \mu(x) + \mu(x) = 2\mu(x)$, so that $\mu(x) \geq 0.5$. This means that $\{x_t : x_t \in \wedge q\} = \emptyset$. Therefore the case $\alpha = \in \wedge q$ in the above two definitions are omitted.

Theorem 3.3. *The support of any non-zero (α, β)-fuzzy ideal μ of R is an ideal of R.*

Proof. Let $x, y \in S_\mu$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Let $\mu(x-y) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\mu(x)} \alpha \mu$ and $y_{\mu(y)} \alpha \mu$ but $\mu(x-y) = 0 < \min\{\mu(x), \mu(y)\}$ and $\mu(x-y) + \min\{\mu(x),$ $\mu(y)\} \leq 0 + 1 = 1$. So $(x-y)_{\min\{\mu(x), \mu(y)\}} \overline{\beta} \mu$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $\mu(x-y) > 0$, that is, $x - y \in S_\mu$. Also $x_1 q \mu$ and $y_1 q \mu$ but $(x-y)_1 \overline{\beta} \mu$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence $\mu(x-y) > 0$, that is, $x - y \in S_\mu$. Let $x \in S_\mu$ and $y \in R$. Suppose that $\mu(y+x-y) = 0$ and let $\alpha \in \{\in, \in \vee q\}$. Then $x_{\mu(x)} \alpha \mu$ but $(y+x-y)_{\mu(x)} \overline{\beta} \mu$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Also $x_1 q \mu$ but $(y+x-y)_1 \overline{\beta} \mu$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence $\mu(y+x-y) > 0$, that is, $y + x - y \in S_\mu$. Let $x \in R$ and $y \in S_\mu$. Assume that $\mu(xy) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $y_{\mu(y)} \alpha \mu$ but $(xy)_{\mu(y)} \overline{\beta} \mu$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. This is a contradiction. Also $y_1 q \mu$ but $(xy)_1 \overline{\beta} \mu$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Therefore $\mu(xy) > 0$ and so $xy \in S_\mu$. Similarly, $(x+z)y - xy \in S_\mu$. Hence S_μ is an ideal of R. □

Definition 3.4. A fuzzy subset μ of R is said to be an $(\in, \in \vee q)$-fuzzy subnear-ring of R if for all $x, y \in R$ and $t, r \in (0,1]$:

(1) $x_t, y_r \in \mu$ implies $(x+y)_{\min\{t,r\}} \in \vee q \mu$,

(2) $x_t \in \mu$ implies $(-x)_t \in \vee q \mu$,

(3) $x_t, y_r \in \mu$ implies $(xy)_{\min\{t,r\}} \in \vee q \mu$.

Theorem 3.5.[22] *A fuzzy subset μ of R is an $(\in, \in \vee q)$-fuzzy subnear-ring of R if and only if $\mu(x-y), \mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$, for all $x, y \in R$.* □

Definition 3.6 [22] A fuzzy subset μ of R is said to be an $(\in, \in \vee q)$-fuzzy ideal of R if for all $x, y, z \in R$ and $t, r \in (0,1]$:

(1) $x_t, y_r \in \mu$ implies $(x-y)_{\min\{t,r\}} \in \vee q\mu$,

(2) $x_t \in \mu$ and $y \in R$ implies $(y+x-y)_t \in \vee q\mu$,

(3) $y_t \in \mu$ and $x \in R$ implies $(xy)_t \in \vee q\mu$,

(4) $z_t \in \mu$ and $x, y \in R$ implies $((x+z)y - xy)_t \in \vee q\mu$.

A fuzzy subset μ which is an $(\in, \in \vee q)$-fuzzy left ideal of R if it satisfies (1),(2) and (3). A fuzzy subset μ which is an $(\in, \in \vee q)$-fuzzy right ideal of R if it satisfies (1),(2) and (4).

Theorem 3.7.[22] *A fuzzy subset μ of R is an $(\in, \in \vee q)$-fuzzy ideal of R if and only if for all $x, y, z \in R$:*

(1) $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$,

(2) $\mu(y+x-y) \geq \min\{\mu(x), 0.5\}$,

(3) $\mu(xy) \geq \min\{\mu(y), 0.5\}$,

(4) $\mu((x+z)y - xy) \geq \min\{\mu(z), 0.5\}$. □

Theorem 3.8. *Let I be an ideal of R. For every $t \in (0, 0.5]$, there exists an $(\in, \in \vee q)$-fuzzy ideal μ of R such that $\mu_t = I$.*

Proof. Let μ be a fuzzy subset in R defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in R$, where $t \in (0, 0.5]$. Obviously, $\mu_t = I$. Assume that $\mu(x-y) < \min\{\mu(x), \mu(y), 0.5\}$, for some $x, y \in R$. Since μ has two valued, that is $\mid Im(\mu) \mid = 2$ it follows that $\mu(x-y) = 0$ and $\min\{\mu(x), \mu(y), 0.5\} = t$. Hence $\mu(x) = t = \mu(y)$ and so $x, y \in I$. Thus $x - y \in I$, since I is an ideal of R and so $\mu(x-y) = t$, which is a contradiction. Therefore $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$. Let us suppose that $\mu(y+x-y) < \min\{\mu(x), 0.5\}$, for some $x, y \in R$. It follows that $\mu(y+x-y) = 0$ and $\min\{\mu(x), 0.5\} = t$. Hence $\mu(x) = t$ and so $x \in I$. Since I is an ideal of R, then $y + x - y \in I$. Thus $\mu(y+x-y) = t$, which is a contradiction and hence $\mu(y+x-y) \geq \min\{\mu(x), 0.5\}$. If there exist $x, y \in R$ such that $\mu(xy) < \min\{\mu(y), 0.5\}$. Then $\mu(xy) = 0$ and $\min\{\mu(y), 0.5\} = t$ and so $\mu(y) = t$. This implies that $y \in I$. Since I is an ideal of R, then $xy \in I$. Thus $\mu(xy) = t$. This leads to a contradiction, which implies that $\mu(xy) \geq \min\{\mu(y), 0.5\}$. Similarly, the same procedure we have $\mu((x+z)y - xy) \geq \min\{\mu(z), 0.5\}$. □

Theorem 3.9. *If I is an ideal of R, then a fuzzy subset μ of R such that*

$$\mu(x) = \begin{cases} \geq 0.5 & \textit{if } x \in I \\ 0 & \textit{otherwise} \end{cases}$$

is an $(\alpha, \in \vee q)$-fuzzy ideal of R.

Proof. (a) Let $x, y \in R$ and $t, r \in (0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Thus

$x, y \in I$ and so $x - y \in I$, that is, $\mu(x-y) \geq 0.5$. If $\min\{t,r\} \leq 0.5$, than $\mu(x-y) \geq 0.5 \geq \min\{t,r\}$. Hence $(x-y)_{\min\{t,r\}} \in \mu$. If $\min\{t,r\} > 0.5$, than $\mu(x-y) + \min\{t,r\} > 0.5 + 0.5 = 1$ and so $(x-y)_{\min\{t,r\}} q\mu$. Therefore $(x-y)_{\min\{t,r\}} \in \vee q\mu$. Now, let $x, y \in R$ and $t \in (0,1]$ be such that $x_t \in \mu$. Then $\mu(x) \geq t$, which implies $x \in I$ and so $(y + x - y) \in I$. Consequently $\mu(y+x-y) \geq 0.5$. If $t \leq 0.5$, then $\mu(y+x-y) \geq 0.5 \geq t$. Hence $(y+x-y)_t \in \mu$. If $t > 0.5$, then $\mu(y+x-y) + t > 0.5 + 0.5 = 1$ and so $(y+x-y)_t q\mu$. Thus $(y+x-y)_t \in \vee q\mu$. Also, let $x, y \in R$ and $t \in (0,1]$ be such that $y_t \in \mu$. Then $\mu(y) \geq t$. Thus $y \in I$ and so $xy \in I$, that is, $\mu(xy) \geq 0.5$. If $t \leq 0.5$, then $\mu(xy) \geq 0.5 \geq t$. Hence $(xy)_t \in \mu$. If $t > 0.5$, then $\mu(xy) + t > 0.5 + 0.5 = 1$ and so $(xy)_t q\mu$. This implies that $(xy)_t \in \vee q\mu$. Similarly, we can prove that $((x+z)y - xy)_t \in \vee q\mu$. Therefore μ is an $(\in, \in \vee q)$-fuzzy ideal of R.

(b) Let $x, y \in R$ and $t, r \in (0,1]$ be such that $x_t q\mu$ and $y_r q\mu$. Then $\mu(x) + t > 1$ and $\mu(y) + r > 1$ implies $x, y \in I$. Since $x - y \in I$, we have $\mu(x-y) \geq 0.5$. If $\min\{t,r\} \leq 0.5$, then $\mu(x-y) \geq 0.5 \geq \min\{t,r\}$. Hence $(x-y)_{\min\{t,r\}} \in \mu$. If $\min\{t,r\} > 0.5$, then $\mu(x-y) + \min\{t,r\} > 0.5 + 0.5 = 1$ and so $(x-y)_{\min\{t,r\}} q\mu$. Thus $(x-y)_{\min\{t,r\}} \in \vee q\mu$. Now, let $x, y \in R$ and $t \in (0,1]$ be such that $x_t q\mu$. This means that $\mu(x) + t > 1$. Thus $x \in I$ and so $y + x - y \in I$, since I is an ideal of R. This implies that $\mu(y+x-y) \geq 0.5$. If $t \leq 0.5$, then $\mu(y+x-y) \geq 0.5 \geq t$. Hence $(y+x-y)_t \in \mu$. If $t > 0.5$, then $\mu(y+x-y) + t > 0.5 + 0.5 = 1$ and so $(y+x-y)_t q\mu$. Thus $(y+x-y)_t \in \vee q\mu$. Again, let $x, y \in R$ and $t \in (0,1]$ be such that $y_t q\mu$. Then $\mu(y) + t > 1$ implies $y \in I$. So $xy \in I$, which implies that $\mu(xy) \geq 0.5$. If $t \leq 0.5$, then $\mu(xy) \geq 0.5 \geq t$. Hence $(xy)_t \in \mu$. If $t > 0.5$, then $\mu(xy) + t > 0.5 + 0.5 = 1$ and so $(xy)_t q\mu$. Thus $(xy)_t \in \vee q\mu$. Let $x, y, z \in R$ and $t \in (0,1]$ be such that $z_t q\mu$. Then $\mu(z) + t > 1$ and it follows that $z \in I$. Then $(x+z)y - xy \in I$ and so $\mu((x+z)y - xy) \geq 0.5$. If $t \leq 0.5$, then $\mu((x+z)y - xy) \geq 0.5 \geq t$. Hence $((x+z)y - xy)_t \in \mu$. If $t > 0.5$, then $\mu((x+z)y - xy) + t > 0.5 + 0.5 = 1$ and so $((x+z)y - xy)_t q\mu$. Thus $((x+z)y - xy)_t \in \vee q\mu$. Hence $(q, \in \vee q)$-fuzzy ideal of R.

(c) Similar consequence of (a) and (b), we have to prove that $(\in \vee q, \in \vee q)$-fuzzy ideal of R. □

Theorem 3.10. *Every $(\in \vee q, \in \vee q)$-fuzzy ideal of R is an $(\in, \in \vee q)$-fuzzy ideal of R.*

Proof. Let μ be an $(\in \vee q, \in \vee q)$-fuzzy ideal of R. Suppose that $x, y \in R$ and $t, r \in (0,1]$ such that $x_t \in \mu$ and $y_r \in \mu$. Then $x_t \in \vee q\mu$ and $y_r \in \vee q\mu$. By the hypothesis $(x-y)_{\min\{t,r\}} \in \vee q\mu$. Now, let $x, y \in R$ and $t \in (0,1]$ such that $x_t \in \mu$. Then $x_t \in \vee q\mu$, so by hypothesis $(y+x-y)_t \in \vee q\mu$. For $x, y \in R$ and $t \in (0,1]$. Then $y_t \in \mu$ implies $y_t \in \vee q\mu$. Thus $(xy)_t \in \vee q\mu$, because μ be an $(\in \vee q, \in \vee q)$-fuzzy ideal of R. Similarly, we prove $((x+z)y - xy)_t \in \vee q\mu$. Therefore μ is an $(\in, \in \vee q)$-fuzzy ideal of R. □

Theorem 3.11. *A fuzzy subset μ of R is an $(\in, \in)$-fuzzy ideal of R if and only if it is a fuzzy ideal of R.*

Proof. Assume that μ is a fuzzy ideal of R. Let $x, y \in R$ and $t, r \in (0,1]$ be such that $x_t, y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Since μ is a fuzzy subgroup of R, we have $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t,r\}$, it follows that $(x-y)_{\min\{t,r\}} \in \mu$. Now let $x, y \in R$ and $t \in (0,1]$. Then $x_t \in \mu$ and so $\mu(x) \geq t$. Since μ is a fuzzy ideal of R, we have $\mu(y+x-y) \geq \mu(x) \geq t$. Hence $(y+x-y)_t \in \mu$. Again, let $x, y \in R$ and $t \in (0,1]$ be such that $y_t \in \mu$. Then $\mu(y) \geq t$. Since μ is a fuzzy ideal of R, then we have $\mu(xy) \geq \mu(y) \geq t$. Thus $(xy)_t \in \mu$. Again let $x, z, y \in R$ and $t \in (0,1]$ be such that $\mu(z) \geq t$. Since μ is a fuzzy ideal of R, then $\mu((x+z)y - xy) \geq \mu(z) \geq t$. Thus $((x+z)y - xy)_t \in \mu$ and therefore μ is an $(\in, \in)$-fuzzy ideal of R.

Conversely, assume that μ is an $(\in, \in)$-fuzzy ideal of R. On the contrary assume that there exist $x, y \in R$ such that $\mu(x-y) < \min\{\mu(x),\mu(y)\}$. Choose $t \in (0,1]$ such that $\mu(x-y) < t < \min\{\mu(x),\mu(y)\}$. Then $x_t, y_t \in \mu$ and $(x-y)_t \overline{\in} \mu$. This is a contradiction to our assumption that μ is an $(\in,\in)$-fuzzy ideal of R. Thus $\mu(x-y) \geq \min\{\mu(x),\mu(y)\}$. Suppose that $\mu(y+x-y) < \mu(x)$, for some $x,y \in R$. Choose $t \in (0,1]$ such that $\mu(y+x-y) < t < \mu(x)$. Then $x_t \in \mu$ and $(y+x-y)_t \overline{\in} \mu$, which is a contradiction and hence $\mu(y+x-y) \geq \mu(x)$. Let us assume that $\mu(xy) < \mu(y)$, for some $x,y \in R$. Then there exist $t \in (0,1]$ such that $\mu(xy) < t < \mu(y)$. This implies that $y_t \in \mu$ but $(xy)_t \overline{\in} \mu$. This contradicts our hypothesis. Hence $\mu(xy) \geq \mu(y)$. Again the contrary assume that there exist $x,y,z \in R$ such that $\mu((x+z)y-xy) < \mu(z)$. Let $t \in (0,1]$ be such that $\mu((x+z)y-xy) < t < \mu(z)$. Then $z_t \in \mu$ but $((x+z)y-xy)_t \overline{\in} \mu$, which is a contradiction and so $\mu((x+z)y-xy) \geq \mu(z)$. Therefore μ is a fuzzy ideal of R. □

Theorem 3.12. *Every $(\in,\in)$-fuzzy ideal of R is an $(\in,\in\vee q)$-fuzzy ideal of R.*

Proof. The proof is straightforward. □

The converse of above theorem is not true in general as show in following example.

Example 3.13 Let $R = \{a,b,c,d\}$ be a set with two binary operations defined as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(R,+,\cdot)$ is a near-ring. Define a fuzzy subset $\mu : R \longrightarrow [0,1]$ of R as follows $\mu(a) = 0.8, \mu(b) = 0.6$ and $\mu(c) = 0.5 = \mu(d)$. Then, clearly, μ is an $(\in,\in\vee q)$-fuzzy ideal of R. But μ is not an $(\in,\in)$-fuzzy ideal of R, since $b_{0.58} \in \mu$ but $((c+b)d-cd)_{0.58} = (d)_{0.58} \overline{\in} \mu$.

In the following theorem, we give a condition for an $(\in,\in\vee q)$-fuzzy ideal of R to be an $(\in,\in)$-fuzzy ideal of R.

Theorem 3.14. *Let μ be an $(\in,\in\vee q)$-fuzzy ideal of R such that $\mu(x) < 0.5$ for all $x \in R$. Then μ is an $(\in,\in)$-fuzzy ideal of R.*

Proof. Let $x,y \in R$ and $t,r \in (0,1]$ be such that $x_t, y_r \in \mu$. Then $\mu(x) \geq t, \mu(y) \geq r$. Since μ is an $(\in,\in\vee q)$-fuzzy subgroup of R, then $\mu(x-y) \geq \min\{\mu(x),\mu(y),0.5\} \geq \min\{t,r,0.5\} = \min\{t,r\}$ and so $(x-y)_{\min\{t,r\}} \in \mu$. Let $x,y \in R$ and $t \in (0,1]$. Then $x_t \in \mu$ and so $\mu(x) \geq t$. Thus $\mu(y+x-y) \geq \min\{\mu(x),0.5\} \geq t$, since μ is an $(\in,\in\vee q)$-fuzzy ideal of R. Hence $(y+x-y)_t \in \mu$. For $x,y \in R$ and $t \in (0,1]$ be such that $y_t \in \mu$. Then $\mu(y) \geq t$. Thus $\mu(xy) \geq \min\{\mu(y),0.5\} \geq t$, since μ is an $(\in,\in\vee q)$-fuzzy ideal of R. Therefore $(xy)_t \in \mu$. Similarly, we can prove that $((x+z)y-xy)_t \in \mu$. Therefore μ is an $(\in,\in)$-fuzzy ideal of R. □

4 $(\in,\in\vee q_k)$-fuzzy subnear-ring and ideals

In this section, we discussed some characterizations of $(\in,\in\vee q_k)$-fuzzy ideals of near-rings.

In [28, 21], defines the following:

1. $x_t q_k \mu$ if $\mu(x) + t + k > 1$.
2. $x_t \in \vee q_k \mu$ if $x_t \in \mu$ or $x_t q_k \mu$.
3. $x_t \in \wedge q_k \mu$ if $x_t \in \mu$ and $x_t q_k \mu$.
4. $x_t \overline{\alpha} \mu$ if $x_t \alpha \mu$ does not hold for $\alpha \in \{q_k, \in \vee q_k, \in \wedge q_k\}$, where $k \in [0, 1)$.

Definition 4.1.[14] A fuzzy subset μ of R is said to be an $(\in, \in \vee q_k)$-fuzzy subnear-ring of R if for all $x, y \in R$ and $t, r \in (0, 1]$:

(1) $x_t, y_r \in \mu$ implies $(x + y)_{\min\{t,r\}} \in \vee q_k \mu$,

(2) $x_t \in \mu$ implies $(-x)_t \in \vee q_k \mu$,

(3) $x_t, y_r \in \mu$ implies $(xy)_{\min\{t,r\}} \in \vee q_k \mu$.

Theorem 4.2.[14] *A fuzzy subset μ of R is an $(\in, \in \vee q_k)$-fuzzy subnear-ring of R if and only if $\mu(x - y), \mu(xy) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$, for all $x, y \in R$.*

Definition 4.3[14] A fuzzy subset μ of R is said to be an $(\in, \in \vee q_k)$-fuzzy ideal of R if for all $x, y, z \in R$ and $t, r \in (0, 1]$:

(1) $x_t, y_r \in \mu$ implies $(x - y)_{\min\{t,r\}} \in \vee q_k \mu$,

(2) $x_t \in \mu$ implies $(y + x - y)_t \in \vee q_k \mu$,

(3) $y_t \in \mu$ and $x \in R$ implies $(xy)_t \in \vee q_k \mu$,

(4) $z_t \in \mu$ and $x, y \in R$ implies $((x + z)y - xy)_t \in \vee q_k \mu$.

A fuzzy subset μ which is an $(\in, \in \vee q_k)$-fuzzy left ideal of R if it satisfies (1),(2) and (3). A fuzzy subset μ which is an $(\in, \in \vee q_k)$-fuzzy right ideal of R if it satisfies (1),(2) and (4).

Theorem 4.4.[14] *A fuzzy subset μ of R is an $(\in, \in \vee q_k)$-fuzzy ideal of R if and only if for all $x, y, z \in R$:*

(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$,

(2) $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\}$,

(3) $\mu(xy) \geq \min\{\mu(y), \frac{1-k}{2}\}$,

(4) $\mu((x + z)y - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$.

Theorem 4.5. *The support of any non-zero $(\in, \in \vee q_k)$-fuzzy ideal(subnear-ring) μ of R is an ideal of R.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy ideal of R. Let $x, y \in S_\mu$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Since μ is an $(\in, \in \vee q_k)$-fuzzy subnear-ring of R, we have $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} > 0$. Let $x \in S_\mu$ and $y \in R$. By Theorem 11, we have $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\} > 0$ and so $y + x - y \in S_\mu$. Let $y \in S_\mu$ and $x \in R$. Now by theorem, we have $\mu(xy) \geq \min\{\mu(y), \frac{1-k}{2}\} > 0$, so $xy \in S_\mu$. Again let $z \in S_\mu$ and $x, y \in R$ be such that $\mu(z) > 0$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, then $\mu((x + z)y - xy) \geq \min\{\mu(z), \frac{1-k}{2}\} > 0$ and hence $(x + z)y - xy \in S_\mu$. Therefore S_μ is an ideal of R. □

Theorem 4.6. *Let I be an ideal(subnear-ring) of R. For every $t \in (0, \frac{1-k}{2}]$, there exists an $(\in, \in \vee q_k)$-fuzzy ideal(subnear-ring) of R such that $\mu_t = I$.*

Proof. Let μ be a fuzzy subset in R defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in R$, where $t \in (0, \frac{1-k}{2}]$. Obviously, $\mu_t = I$. Suppose that $\mu(x-y) < \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$, for some $x, y \in R$. Then $\mu(x-y) = 0$ and $\min\{\mu(x), \mu(y), \frac{1-k}{2}\} = t$, since $|Im(\mu)| = 2$. Hence $\mu(x) = t = \mu(y)$ and so $x, y \in I$. Since I is an ideal of R, then we have $x - y \in I$, and so $\mu(x-y) = t$. This is a contradiction. Therefore $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$. Let us suppose that $\mu(y+x-y) < \min\{\mu(x), \frac{1-k}{2}\}$, for some $x, y \in R$. It follows that $\mu(y+x-y) = 0$ and $\min\{\mu(x), \frac{1-k}{2}\} = t$. Hence $\mu(x) = t$ and so $x \in I$. Since I is an ideal of R, then $y + x - y \in I$. Thus $\mu(y+x-y) = t$, which is a contradiction and hence $\mu(y+x-y) \geq \min\{\mu(x), \frac{1-k}{2}\}$. If there exist $x, y \in R$ such that $\mu(xy) < \min\{\mu(y), \frac{1-k}{2}\}$. Then $\mu(xy) = 0$ and $\min\{\mu(y), \frac{1-k}{2}\} = t$. Thus $\mu(y) = t$ and so $y \in I$. Since I is an ideal of R, then $xy \in I$. Thus $\mu(xy) = t$, which is a contradiction and hence $\mu(xy) \geq \min\{\mu(y), \frac{1-k}{2}\}$. Similarly, the same procedure we have $\mu((x+z)y - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$. □

Theorem 4.7. *If I is an ideal(subnear-ring) of R, then a fuzzy subset μ of R such that*

$$\mu(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

is an $(\alpha, \in \vee q_k)$-fuzzy ideal(subnear-ring) of R, where $\alpha \in \{\in, q, \in \vee q, q_k, \in \vee q_k\}$.

Proof. (a) Let $x, y \in R$ and $t, r \in (0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t > 0$ and $\mu(y) \geq r > 0$. This implies that $\mu(x) \geq \frac{1-k}{2}$ and $\mu(y) \geq \frac{1-k}{2}$. Thus $x, y \in I$ and so $x - y \in I$, that is, $\mu(x-y) \geq \frac{1-k}{2}$. If $\min\{t, r\} \leq \frac{1-k}{2}$, than $\mu(x-y) \geq \frac{1-k}{2} \geq \min\{t, r\}$. Hence $(x-y)_{\min\{t,r\}} \in \mu$. If $\min\{t, r\} > \frac{1-k}{2}$, than $\mu(x-y) + \min\{t, r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(x-y)_{\min\{t,r\}} q_k \mu$. Therefore $(x-y)_{\min\{t,r\}} \in \vee q_k \mu$. Now let $x, y \in R$ and $t \in (0,1]$ be such that $x_t \in \mu$. Then $\mu(x) \geq t$, which implies $x \in I$ and so $(y + x - y) \in I$. Consequently $\mu(y+x-y) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $\mu(y+x-y) \geq \frac{1-k}{2} \geq t$. Hence $(y+x-y)_t \in \mu$. If $t > \frac{1-k}{2}$, then $\mu(y+x-y) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(y+x-y)_t q_k \mu$. Thus $(y+x-y)_t \in \vee q_k \mu$. Also let $x, y \in R$ and $t \in (0,1]$ be such that $y_t \in \mu$. Then $\mu(y) \geq t$. Thus $y \in I$ and so $xy \in I$, that is $\mu(xy) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $\mu(xy) \geq \frac{1-k}{2} \geq t$. Hence $(xy)_t \in \mu$. If $t > \frac{1-k}{2}$, then $\mu(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(xy)_t q_k \mu$. This implies that $(xy)_t \in \vee q_k \mu$. Similarly, we can prove that $((x+z)y - xy)_t \in \vee q_k \mu$. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R.

(b) Let $x, y \in R$ and $t, r \in (0,1]$ be such that $x_t q \mu$ and $y_r q \mu$. Then $x, y \in I, \mu(x) + t > 1$ and $\mu(y) + r > 1$. Since $x - y \in I$, we have $\mu(x-y) \geq \frac{1-k}{2}$. If $\min\{t, r\} \leq \frac{1-k}{2}$, then $\mu(x-y) \geq \frac{1-k}{2} \geq \min\{t, r\}$. Hence $(x-y)_{\min\{t,r\}} \in \mu$. If $\min\{t, r\} > \frac{1-k}{2}$, then $\mu(x-y) + \min\{t, r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(x-y)_{\min\{t,r\}} q_k \mu$. Thus $(x-y)_{\min\{t,r\}} \in \vee q_k \mu$. Now let $x, y \in R$ and $t \in (0,1]$ be such that $x_t q \mu$. This means that $\mu(x) + t > 1$. Thus $x \in I$ and so $y + x - y \in I$. This implies that $\mu(y+x-y) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $\mu(y+x-y) \geq \frac{1-k}{2} \geq t$. Hence $(y+x-y)_t \in \mu$. If $t > \frac{1-k}{2}$, then $\mu(y+x-y) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(y+x-y)_t q_k \mu$. Thus $(y+x-y)_t \in \vee q_k \mu$. Similarly, we prove the same procedure $(xy)_t \in \vee q_k \mu$. Let $x, y, z \in R$ and $t \in (0,1]$ be such that $z_t q_k \mu$. Then $\mu(z) + t > 1$ and it follows that $z \in I$. Then $((x+z)y - xy) \in I$ and so $\mu((x+z)y - xy) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $\mu((x+z)y - xy) \geq \frac{1-k}{2} \geq t$. Hence $((x+z)y - xy)_t \in \mu$. If $t > \frac{1-k}{2}$, then $\mu((x+z)y - xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $((x+z)y - xy)_t q_k \mu$. Thus $((x+z)y - xy)_t \in \vee q_k \mu$. Hence $(q, \in \vee q_k)$-fuzzy ideal of R.

(c)Similar consequence of (a) and (b), we have to prove that $(\in \vee q, \in \vee q_k)$-fuzzy ideal of R.

(d) Let $x, y \in R$ and $t, r \in (0,1]$ be such that $x_t q_k \mu$ and $y_r q_k \mu$. Then $x, y \in I, \mu(x) + t + k > 1$ and $\mu(y) + r + k > 1$. Since $x - y \in I$, we have $\mu(x-y) \geq \frac{1-k}{2}$. If $\min\{t, r\} \leq \frac{1-k}{2}$, then $\mu(x-y) \geq \frac{1-k}{2} \geq \min\{t, r\}$.

Hence $(x-y)_{\min\{t,r\}} \in \mu$. If $\min\{t,r\} > \frac{1-k}{2}$, then $\mu(x-y) + \min\{t,r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(x-y)_{\min\{t,r\}} q_k \mu$. Thus $(x-y)_{\min\{t,r\}} \in \vee q_k \mu$. Now let $x, y \in R$ and $t \in (0,1]$ be such that $x_t q_k \mu$. This means that $\mu(x) + t + k > 1$. Thus $x \in I$ and so $y + x - y \in I$. This implies that $\mu(y+x-y) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $\mu(y+x-y) \geq \frac{1-k}{2} \geq t$. Hence $(y+x-y)_t \in \mu$. If $t > \frac{1-k}{2}$, then $\mu(y+x-y) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(y+x-y)_t q_k \mu$. Thus $(y+x-y)_t \in \vee q_k \mu$. Similarly, we prove the same procedure $(xy)_t \in \vee q_k \mu$. Let $x, y, z \in R$ and $t \in (0,1]$ be such that $z_t q_k \mu$. Then $\mu(z) + t + k > 1$, and it follows that $z \in I$. Then $((x+z)y - xy) \in I$ and so $\mu((x+z)y - xy) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $\mu((x+z)y - xy) \geq \frac{1-k}{2} \geq t$. Hence $((x+z)y - xy)_t \in \mu$. If $t > \frac{1-k}{2}$, then $\mu((x+z)y - xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $((x+z)y - xy)_t q_k \mu$. Thus $((x+z)y - xy)_t \in \vee q_k \mu$. Hence $(q_k, \in \vee q_k)$-fuzzy ideal of R.

(e) Similar consequence of (a) and (d), we have to prove that $(\in \vee q_k, \in \vee q_k)$-fuzzy ideal of R. □

Theorem 4.8. *Every $(\in \vee q_k, \in \vee q_k)$-fuzzy ideal of R is an $(\in, \in \vee q_k)$-fuzzy ideal of R.*

Proof. Let μ be an $(\in \vee q_k, \in \vee q_k)$-fuzzy ideal of R. Suppose that $x, y \in R$ and $t, r \in (0,1]$ such that $x_t \in \mu$ and $y_r \in \mu$. Then $x_t \in \vee q_k \mu$ and $y_r \in \vee q_k \mu$. By the hypothesis $(x-y)_{\min\{t,r\}} \in \vee q_k \mu$. Now $x, y \in R$ and $t \in (0,1]$ such that $x_t \in \mu$. Then $x_t \in \vee q_k \mu$, so by hypothesis $(y+x-y)_t \in \vee q_k \mu$. Similarly, we prove $((x+z)y - xy)_t \in \vee q_k \mu, (xy)_t \in \vee q_k \mu$. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R. □

Theorem 4.9. *Every $(\in, \in)$-fuzzy ideal of R is an $(\in, \in \vee q_k)$-fuzzy ideal of R.*

Proof. The proof is straightforward. □

In the following theorem, we give a condition for an $(\in, \in \vee q_k)$-fuzzy ideal of R to be an $(\in, \in)$-fuzzy ideal of R.

Theorem 4.10. *Let μ be an $(\in, \in \vee q_k)$-fuzzy ideal of R such that $\mu(x) < \frac{1-k}{2}$ for all $x \in R$. Then μ is an $(\in, \in)$-fuzzy ideal of R.*

Proof. Let $x, y \in R$ and $t, r \in (0,1]$ be such that $x_t, y_r \in \mu$. Then $\mu(x) \geq t, \mu(y) \geq r$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, then $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min\{t, r, \frac{1-k}{2}\} = \min\{t, r\}$ and so $(x-y)_{\min\{t,r\}} \in \mu$. Let $x, y \in R$ and $t \in (0,1]$ be such that $x_t \in \mu$. Then $\mu(x) \geq t$. Thus $\mu(y+x-y) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq t$, since μ is an $(\in, \in \vee q)$-fuzzy ideal of R. Hence $(y+x-y)_t \in \mu$. Let $x, y \in R$ and $t \in (0,1]$ be such that $y_t \in \mu$. Then $\mu(y) \geq t$ and so $\mu(xy) \geq \min\{\mu(y), \frac{1-k}{2}\} \geq t$, since μ is an $(\in, \in \vee q)$-fuzzy ideal of R. Thus $(xy)_t \in \mu$. Similarly, we can prove that $((x+z)y - xy)_t \in \mu$. Therefore μ is an $(\in, \in)$-fuzzy ideal of R. □

Definition 4.11 For any fuzzy subset μ of R and $t \in (0,1]$, we consider two subsets: $Q_k(\mu; t) = \{x \in R \mid x_t q_k \mu\}$ and $[\mu_k]_t = \{x \in R \mid x_t \in \vee q_k \mu\}$. Obviously, $[\mu_k]_t = \mu_t \cup Q_k(\mu; t)$.

We call $[\mu_k]_t$ is $(\in \vee q_k)$-level ideal and $Q_k(\mu_k; t)$ a q_k-level ideal of μ.

Theorem 4.12. *Every fuzzy subset μ of R satisfies the following assertion $t \in (0, \frac{1-k}{2}] \Rightarrow [\mu_k]_t = \mu_t$.*

Proof. Let $t \in (0, \frac{1-k}{2}]$. Clearly, $\mu_t \subseteq [\mu_k]_t$. Let $x \in [\mu_k]_t$. If $x \notin \mu_t$, then $\mu(x) < 1$ and so $\mu(x) + t + k \leq t + t + k = 2t + k \leq 1$. This implies that $x_t \overline{q}_k \mu$, that is, $x \notin Q_k(\mu; t)$. Thus $x \notin \mu_t \cup Q_k(\mu; t) = [\mu_k]_t$. This leads to a contradiction and so $x \in \mu$. Thus $[\mu_k]_t \subseteq \mu_t$. Therefore $[\mu_k]_t = \mu_t$. □

Theorem 4.13. *A fuzzy subset μ of R is an $(\in, \in \vee q_k)$-fuzzy ideal of R if and only if $[\mu_k]_t (\neq \emptyset)$ is an ideal of R.*

Proof. Assume that μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R and let $t \in (0,1]$ be such that $[\mu_k]_t (\neq \emptyset)$. Let $x, y \in [\mu_k]_t$ such that $\mu(x) \geq t$ or $\mu(x) + t + k > 1$ and $\mu(y) \geq t$ or $\mu(y) + t + k > 1$. We can consider four cases:

(i) $\mu(x) \geq t$ and $\mu(y) \geq t$,

(ii) $\mu(x) \geq t$ and $\mu(y) + t + k > 1$,

(iii) $\mu(x) + t + k > 1$ and $\mu(y) \geq t$,

(iv) $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$.

Consider Case (i): $\mu(x) \geq t$ and $\mu(y) \geq t$. This implies that

$$\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} = \begin{cases} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{cases}$$

If $t > \frac{1-k}{2}$, then $\mu(x-y) \geq \frac{1-k}{2}$ and so $\mu(x-y) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1-k$, that is, $(x-y)_t q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(x-y) \geq t$ and thus $(x-y)_t \in \mu$. Therefore, $(x-y)_t \in \vee q_k \mu$, that is, $[\mu_k]_t$.

Case(ii): $\mu(x) \geq t$ and $\mu(y) + t + k > 1$. If $t > \frac{1-k}{2}$, then $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min\{t, 1-k-t, \frac{1-k}{2}\} = 1-k-t$, that is, $\mu(x-y) + t + k > 1$ and thus $(x-y)_t q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min\{t, , 1-k-t, \frac{1-k}{2}\} = t$, that is, $(x-y)_t \in \mu$ and thus $(x-y)_t \in \vee q_k \mu$. This means that $x - y \in [\mu_k]_t$. Similarly, we can prove the result for the case(iii). Next we consider the case(iv): $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$. If $t > \frac{1-k}{2}$, then $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min\{1-k-t, 1-k-t, \frac{1-k}{2}\} = 1-k-t$. So, $\mu(x-y) + t + k > 1$, that is, $(x-y)_t q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min\{1-k-t, 1-k-t, \frac{1-k}{2}\} = \frac{1-k}{2} \geq t$, that is, $(x-y)_t \in \mu$ and hence $(x-y)_t \in \vee q_k \mu$. This means that $x - y \in [\mu_k]_t$. Consequently, $[\mu_k]_t$ is an additive subgroup of $(R, +)$. Let $x \in [\mu_k]_t$ and $y \in R$ such that $\mu(x) \geq t$ and $\mu(x) + t + k > 1$ and we consider two cases:

Case(i): $\mu(x) \geq t$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, we have $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$, then $\mu(y + x - y) \geq \frac{1-k}{2}$ and so $\mu(y + x - y) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1-k$, that is, $\mu(y + x - y) + t + k > 1$. Thus $(y + x - y)_t q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(y + x - y) \geq t$. Hence $(y + x - y)_t \in \mu$.

Case(ii): $\mu(x) + t + k > 1$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, we have $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\} > \min\{1-k-t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$, then $\mu(y + x - y) > 1-k-t$. Thus $(y + x - y)_t q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(y + x - y) > \frac{1-k}{2} \geq t$. Hence $(y + x - y)_t \in \mu$. This means that $(y + x - y)_t \in \vee q_k \mu$, that is, $y + x - y \in [\mu_k]_t$ and therefore $[\mu_k]_t$ is an ideal of R. Let $y \in [\mu_k]_t$ and $x \in R$ such that $\mu(y) \geq t$ and $\mu(y) + t + k > 1$. We consider two cases:

Case(i): $\mu(y) \geq t$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R by Theorem, we have $\mu(xy) \geq \min\{\mu(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$, then $\mu(xy) \geq \frac{1-k}{2}$ and thus $\mu(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1-k$. So, $\mu(xy) + t + k > 1$. This means that $(xy)_t q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(xy) \geq t$. Hence $(xy)_t \in \mu$.

Case(ii): $\mu(y) + t + k > 1$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, we have $\mu(xy) \geq \min\{\mu(y), \frac{1-k}{2}\} > \min\{1-k-t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$, then $\mu(xy) > 1-k-t$. Thus $(xy)_t q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(xy) > \frac{1-k}{2} \geq t$. Hence $(xy)_t \in \mu$. This implies that $(xy)_t \in \vee q_k \mu$, that is, $xy \in [\mu_k]_t$ and therefore $[\mu_k]_t$ is a left ideal of R. Again let $x, y \in R$ and $z \in [\mu_k]_t$ for $0 < t \leq 1$. Then $z_t \in \vee q_k \mu$, that is, $\mu(z) \geq t$ and $\mu(z) + t + k > 1$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, then we have $\mu((x+z)y - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$. Similarly, we can prove that

$(x+z)y - xy \in [\mu_k]_t$ and $[\mu_k]_t$ the right ideal of R. Therefore, $[\mu_k]_t$ is an ideal of R.

Conversely, assume that μ be a fuzzy subset in R and let $t \in (0,1]$ be such that $[\mu_k]_t$ is an ideal of R. Suppose that $\mu(x-y) < \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$. Choose t such that $\mu(x-y) < t < \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$. Then $t \in (0, \frac{1-k}{2}]$ and $x, y \in \mu_t \subseteq [\mu_k]_t$. Since $[\mu_k]_t$ is an ideal of R, then $x - y \in [\mu_k]_t$ and we have $\mu(x-y) \geq t$ or $\mu(x-y) + t + k > 1$, which is a contradiction. Thus $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$, for all $x, y \in R$. Now, let $x, y \in R$ be such that $\mu(y+x-y) < t < \min\{\mu(x), \frac{1-k}{2}\}$. Then $t \in (0, \frac{1-k}{2}]$ and $x \in \mu_t \subseteq [\mu_k]_t$. Since $[\mu_k]_t$ is an ideal of R, then $y + x - y \in [\mu_k]_t$ and so $\mu(y+x-y) \geq t$ or $\mu(y+x-y) + t + k > 1$. This is a contradiction to our assumption. Hence $\mu(y+x-y) \geq \min\{\mu(x), \frac{1-k}{2}\}$, for all $x, y \in R$. If there exist $x, y \in R$ such that $\mu(xy) < t < \min\{\mu(y), \frac{1-k}{2}\}$. Then $t \in (0, \frac{1-k}{2}]$ and $y \in \mu_t \subseteq [\mu_k]_t$. Thus $xy \in [\mu_k]_t$,since $[\mu_k]_t$ is an ideal of R and so $\mu(xy) \geq t$ or $\mu(xy) + t + k > 1$, which is a contradiction. Therefore $\mu(xy) \geq \min\{\mu(y), \frac{1-k}{2}\}$, for all $x, y \in R$. Thus μ is an $(\in, \in \vee q_k)$-fuzzy left ideal of R. Similarly we have to prove $\mu((x+z)y - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$ and μ is an $(\in, \in \vee q_k)$-fuzzy right ideal of R. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R. □

Theorem 4.14. *If μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, then the set $Q_k(\mu; t)(\neq \emptyset)$ is an ideal of R for all $t \in (0.5, 1]$.*

Proof. Assume that μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R and let $t \in (0.5, 1]$ be such that $Q_k(\mu; t) \neq \emptyset$. Let $x, y \in Q_k(\mu; t)$ be such that $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$ and we have $\mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$. If $\min\{\mu(x), \mu(y)\} \geq \frac{1-k}{2}$, then $\mu(x-y) \geq \frac{1-k}{2} > 1 - k - t$. If $\min\{\mu(x), \mu(y)\} < \frac{1-k}{2}$, then $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} > 1 - k - t$. This implies that $x - y \in Q_k(\mu; t)$. Now, let $x \in Q_k(\mu; t)$ and $y \in R$ be such that $\mu(x) + t + k > 1$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, then we have $\mu(y+x-y) \geq \min\{\mu(x), \frac{1-k}{2}\}$. If $\mu(x) \geq \frac{1-k}{2}$, then $\mu(y+x-y) \geq \frac{1-k}{2} > 1 - k - t$. If $\mu(x) < \frac{1-k}{2}$, then $\mu(y+x-y) \geq \mu(x) > 1 - k - t$. Thus $y + x - y \in Q_k(\mu; t)$. Similarly, $xy \in Q_k(\mu; t)$. Again let $x, y \in R$ and $z \in Q_k(\mu; t)$ be such that $\mu(z) + t + k > 1$. Since μ is an $(\in, \in \vee q_k)$-fuzzy ideal of R, then we have $\mu((x+z)y - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$. If $\mu(z) \geq \frac{1-k}{2}$, then $\mu((x+z)y - xy) \geq \frac{1-k}{2} > 1 - k - t$ and if $\mu(z) < \frac{1-k}{2}$, then $\mu((x+z)y - xy) \geq \mu(z) > 1 - k - t$ and thus $(x+z)y - xy \in Q_k(\mu; t)$. Therefore $Q_k(\mu; t)$ is an ideal of R. □

5 A new view of $(\in, \in \vee q_k)$-fuzzy ideals

Let μ and ν be any two fuzzy subset of R. We define a new relation $\subseteq \vee q_k$ on $F(R)$, which is called the fuzzy inclusion or k-quasi-coincidence relation as follows.

For any $\mu, \nu \in F(R)$, by $\mu \subseteq \vee q_k \nu$ we mean that $x_t \in \mu$ implies $x_t \in \vee q_k \nu$ for all $x \in R$ and $t \in (0,1]$ and $\mu \supseteq \vee \overline{q}_k \nu$ we mean that $x_t \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}_k \nu$ for all $x \in R$ and $t \in (0,1]$. Moreover, μ and ν are said to be $(0, \frac{1-k}{2})$-fuzzy equal, denoted by $\mu \approx \nu$ if $\mu \subseteq \vee q_k \nu$ and $\nu \subseteq \vee q_k \mu$. And $\overline{\in \vee q_k}$ means $\in \vee q_k$ does not hold and $\overline{\subseteq \vee q_k}$ implies that $\subseteq \vee q_k$ is not true.

Proposition 5.1. *For any two fuzzy subsets μ and ν of R,*

(i) $\mu \subseteq \vee q_k \nu$ if and only if $\nu(x) \geq \min\{\mu(x), \frac{1-k}{2}\}$, for all $x \in R$,

(i) $\mu \supseteq \vee \overline{q}_k \nu$ if and only if $\max\{\mu(x), \frac{1-k}{2}\} \geq \nu(x)$, for all $x \in R$.

Proof. (i) Let $\mu \subseteq \vee q_k \nu$. Suppose there exist $x \in R$ such that $\nu(x) < t < \min\{\mu(x), \frac{1-k}{2}\}$. Then $x_t \in \mu$, but

$x_t \overline{\subseteq \vee q_k} \nu$, a contradiction. Hence, $\nu(x) \geq \min\{\mu(x), \frac{1-k}{2}\}$.

Conversely, assume that $\nu(x) \geq \min\{\mu(x), \frac{1-k}{2}\}$, for all $x \in R$. If $\mu \overline{\subseteq \vee q_k} \nu$, then there exist $x_t \in \mu$ with $x_t \overline{\in \vee q_k}\ \nu$ and so $\mu(x) \geq t, \nu(x) < t$ and $\nu(x) < \frac{1-k}{2}$, a contradiction.

(ii) Let $\mu \supseteq \vee \overline{q}_k \nu$. If there exist $x \in R$ such that $\nu(x) > t > \min\{\mu(x), \frac{1-k}{2}\}$, then $x_t \in \nu$ with $x_t \overline{\in} \mu$ and $t > \frac{1-k}{2}$. Hence, $x_t \overline{\in} \vee \overline{q}_k \nu$, that is, $x_t \overline{q}_k \nu$, and so, $\nu(x) + t + k \leq 1$. This means that $t \leq \frac{1-k}{2}$, a contradiction.

Conversely, let $\max\{\mu(x), \frac{1-k}{2}\} \geq \nu(x)$, for all $x \in R$. If $\mu \overline{\subseteq \vee \overline{q}_k} \nu$, then there exist $x_t \overline{\in} \mu$ but $x_t \overline{\overline{\in} \vee \overline{q}_k}\ \nu$. Thus $\mu(x) < t, \nu(x) \geq t$ and $\nu(x) + t + k > 1$. If $\mu(x) > \frac{1-k}{2}$, then $\mu(x) \geq \nu(x)$, a contradiction. If $\mu(x) \leq \frac{1-k}{2}$, then $\frac{1-k}{2} \geq \nu(x)$. Since $\nu(x) \geq t$ and $\frac{1-k}{2} \geq \nu(x) \geq t$, then $1 < \nu(x) + t + k \leq 2\nu(x)$. So, $\nu(x) > \frac{1-k}{2}$, a contradiction.
□

Definition 5.2 A fuzzy subset μ of R is called a new $(\in, \in \vee q_k)$-fuzzy subnear-ring of R if it satisfies:

(1) $(\mu + \mu) \subseteq \vee q_k \mu$,

(2) $\mu^{-1} \subseteq \vee q_k \mu$,

(3) $(\mu \circ \mu) \subseteq \vee q_k \mu$.

Definition 5.3 A fuzzy subset μ of R is called a new $(\in, \in \vee q_k)$-fuzzy ideal of R if it is a new $(\in, \in \vee q_k)$-fuzzy subnear-ring of R and if for all $x, y \in R$,

(4) $(x + \mu - x) \subseteq \vee q_k \mu$,

(5) $(\mu \circ \chi_R) \subseteq \vee q_k \mu$,

(6) $(x \circ (y + \mu) - xy) \subseteq \vee q_k \mu$.

Lemma 5.4. *Let μ be a fuzzy subset of R. Then*

1. (a) $(\mu + \mu) \subseteq \vee q_k \mu$ *and*
   (b) $\mu(x + y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$, *for all $x, y \in R$ are equivalent.*
2. (c) $\mu^{-1} \subseteq \vee q_k \mu$ *and*
   (d) $\mu(-x) \geq \min\{\mu(x), \frac{1-k}{2}\}$, *for all $x \in R$ are equivalent.*
3. (e) $(\mu \circ \mu) \subseteq \vee q_k \mu$ *and*
   (f) $\mu(xy) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$, *for all $x, y \in R$ are equivalent.*

Proof. (1) $(a) \Longrightarrow (b)$: (b) is not valid. Suppose that there exist $x, y \in R$ such that $\mu(x+y) < t < \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$, then $t \in \frac{1-k}{2}, x_t, y_t \in \mu$, but $(x + y)_t \overline{\in \vee q_k} \mu$.

$$(\mu + \mu)(x + y) = \sup_{x+y=a+b} \min\{\mu(a), \mu(b)\}$$
$$\geq \min\{\mu(x), \mu(y)\} \geq t.$$

So, $(x + y)_t \in (\mu + \mu)$. Hence, $(x + y)_t \in \vee q_k \mu$, a contradiction. Therefore, $\mu(x + y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$.

$(b) \Longrightarrow (a)$: Let $x_t \in (\mu + \mu)$. If possible, let $\mu(x) < t$ and $\mu(x) < \frac{1-k}{2}$. If $x = a + b$, for some $a, b \in R$, then by hypothesis, (b), we have $\frac{1-k}{2} > \mu(x) = \mu(a + b) \geq \min\{\mu(a), \mu(b), \frac{1-k}{2}\}$. This implies that $\mu(x) \geq \min\{\mu(a), \mu(b)\}$. Hence, we have $t \leq (\mu + \mu)(x) = \sup_{x=a+b} \min\{\mu(a), \mu(b)\} \leq \sup_{x=a+b} \mu(x)$. So, $\mu(x) \geq t$, which contradicts. Therefore, $(\mu + \mu) \subseteq \vee q_k \mu$.

(2) (c) $\Longrightarrow$ (d): (d) is not valid. Suppose that $\mu(-x) < \min\{\mu(x), \frac{1-k}{2}\}$, for all $x, y \in R$. Choose t such that $\mu(-x) < t < \min\{\mu(x), \frac{1-k}{2}\}$. Then, $x_t \in \mu, t \in \frac{1-k}{2}$ and $(-x)_t \overline{\in \vee q_k} \mu$. But $\mu^{-1}(-x) = \mu(x) \geq t$, so, $(-x)_t \mu^{-1}$, which means that $(-x)_t \in \vee q_k \mu$, a contradiction. Thus, $\mu(-x) \geq \min\{\mu(x), \frac{1-k}{2}\}$.

(d) $\Longrightarrow$ (c): Let $x_t \in \mu^{-1}$, but $x_t \overline{\in \vee q_k} \mu$ be such that $\mu(x) < t$ and $\mu(x) < \frac{1-k}{2}$. Then we have $\mu(-x) \geq \min\{\mu(x), \frac{1-k}{2}\} = \mu(x)$ and so $\mu(-x) = \mu(x)$. Thus $t \leq \mu^{-1}(x) = \mu(-x) = \mu(x)$, which is a contradiction. Therefore (d) is hold.

(3) (e)$\Longleftrightarrow$(f). It follows from above (1). □

Theorem 5.5. *A fuzzy subset μ of R is a new $(\in, \in \vee q_k)$-fuzzy subnear-ring of R if and only if it satisfies $\mu(x+y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}, \mu(-x) \geq \min\{\mu(x), \frac{1-k}{2}\}$ and $\mu(xy) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$.* □

The following corollary is a consequence of Lemma 5.4 and Theorem 5.5.

Corollary 5.6. *The concept of new $(\in, \in \vee q_k)$-fuzzy subnear-ring of R and $(\in, \in \vee q_k)$-fuzzy subnear-ring of R are equivalent respectively.* □

Lemma 5.7. *Let μ be a fuzzy subset of R. Then*

1. *(a) μ is a new $(\in, \in \vee q_k)$-fuzzy subnear-ring of R and*
   (b) μ is a new $(\in, \in \vee q_k)$-fuzzy subnear-ring of R are equivalent.
2. *(c) $y + \mu - y \subseteq \vee q_k \mu$, for all $y \in R$ and*
   (d) $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\}$, for all $x, y \in R$ are equivalent.
3. *(e) $f_R \circ \mu \subseteq \vee q_k \mu$ and*
   (f) $\mu(xy) \geq \min\{\mu(x), \frac{1-k}{2}\}$, for all $x, y \in R$ are equivalent.
4. *(g) $(x \circ (y + \mu) - xy \subseteq \vee q_k \mu$, for all $x, y \in R$ and*
   (h) $\mu(x(y+z) - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$, for all $x, y, z \in R$ are equivalent.

Proof. (1) (a)$\Longleftrightarrow$(b). It follows from above Lemma.

(2) (c) $\Longrightarrow$ (d): Suppose that $\mu(y + x - y) < \min\{\mu(x), \frac{1-k}{2}\}$, for all $x, y \in R$. Choose t such that $\mu(y + x - y) < t < \min\{\mu(x), \frac{1-k}{2}\}$. Then $t \in \frac{1-k}{2}, x_t \in \mu$ and $(y + x - y)_t \overline{\in \vee q_k} \mu$. Since $(y + \mu - y)(y + x - y) = \sup_{y+x-y=y+x'-y} \mu(x') \geq \mu(x) \geq t$, we have $(y + x - y) \in (y + \mu - y)$. So, $(y + x - y)_t \in \vee q_k \mu$, a contradiction. Thus $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\}$.

(d) $\Longrightarrow$ (c): Let $x, y \in R$ and $t \in (0, 1]$. Then $x_t \in (y + \mu - y)$, but $x_t \overline{\in \vee q_k} \mu$ and so $\mu(x) < t$ and $\mu(x) < \frac{1-k}{2}$. $(y + \mu - y)(x) = \sup_{x=y+x'-y} \mu(x')$. Since $\frac{1-k}{2} > \mu(x) = \mu(y + x' - y) \geq \min\{\mu(x'), \frac{1-k}{2}\}$, which implies that $\mu(x) \geq \mu(x')$. Hence,
$t \leq (y + \mu - y)(x) \leq \sup_{y+x'-y} \mu(x) = \mu(x)$. So, $\mu(x) \geq t$, which is a contradiction. Therefore $x_t \in \vee q_k \mu$ and hence $(y + \mu - y) \subseteq \vee q_k \mu$.

(3) (e) $\Longrightarrow$ (f): Suppose that $\mu(xy) < \min\{\mu(x), \frac{1-k}{2}\}$, for all $x, y \in R$. Then there exist t such that $\mu(xy) < t < \min\{\mu(x), \frac{1-k}{2}\}$. So, $t \in \frac{1-k}{2}, x_t \in \mu$ and $(xy)_t \overline{\in \vee q_k} \mu$. $(f_R \circ \mu)(xy) = \sup_{xy=x'y} \mu(x') \geq \mu(x) \geq t$. This implies that $(xy) \in (f_R \circ \mu)$, that is, $(xy)_t \in \vee q_k \mu$, a contradiction. Therefore $\mu(xy) \geq \min\{\mu(x), \frac{1-k}{2}\}$.

(f) $\Longrightarrow$ (e): Let $x, y \in R$ and $t \in (0,1]$ be such that $x_t \in (f_R \circ \mu)$, but $x_t \overline{\in \vee q_k} \mu$. Then $\mu(x) < t$ and $\mu(x) < \frac{1-k}{2}$. By definition, $(f_R \circ \mu)(x) = \sup\limits_{x=x'y} \mu(x')$. Thus $\mu(x) \geq \mu(x')$, because $\frac{1-k}{2} > \mu(x) = \mu(x'y) \geq \min\{\mu(x'), \frac{1-k}{2}\}$. So, $t \leq (f_R \circ \mu)(x) \leq \sup\limits_{x=x'y} \mu(x) = \mu(x)$, that is, $\mu(x) \geq t$. This leads to a contradiction. Therefore $(f_R \circ \mu) \subseteq \vee q_k \mu$.

(4)(g)$\Longleftrightarrow$(h). It follows from (2) and (3). □

Theorem 5.8. *A fuzzy subset μ of R is a new $(\in, \in \vee q_k)$-fuzzy ideal of R if and only if it satisfies*

1. *μ is an $(\in, \in \vee q_k)$ fuzzy subnear-ring of R,*
2. $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\}$,
3. $\mu(xy) \geq \min\{\mu(x), \frac{1-k}{2}\}$,
4. $\mu(x(y+z)xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$,

□

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