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A New Approach to Constructing Extended Exponential General Linear Methods for Initial Value Problems in Ordinary Differential Equations

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ABSTRACT. This paper focuses on a new extension approach for the construction of Exponential General Linear Methods. These methods are related to Butcher, Calvo and Palencia and Osisiogu and Bazuaye Methods, but, in contrast to the later, we make use of higher terms of the exponential and related matrix functions within the numerical solution and not in the internal stage order extension as done by Osisiogu and Bazuaye. This feature enables us to derive the order conditions which in turn aided in the construction of family of methods of higher order. The stability behavior is consistent with the existing methods but with the advantage of having less computational effort. Numerical experiments indicate that Extended Exponential General Linear Methods constructed via this approach compete favourably with the existing Methods.

1 Introduction

We consider the numerical solution of certain class autonomous initial value problems in ordinary differential equation(ODEs) of the form

$$y'(t) = Wy(t)) + N(y(t)), 0 \leq t \leq T, \text{ Given } y(0) \tag{1.1}$$

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The development of numerical integrators for this class of problem (1.1) has attracted considerable interest . The reason for this interest cannot be far fetched. Mathematical modelling of physical situations in areas such as electricity,Biological systems can be modelled into a system of ordinary differential equation with enviable stability properties.

Most Numerical methods for solving ordinary differential equations generally fall into two main classes: linear multistep (multivalue) and Runge-Kutta (multistage) methods. Both of these classes have well known advantages and disadvantages. [1] introduced general linear methods as a unifying approach for the traditional methods to study the properties of consistency and convergence and to formulate new methods with clear advantages over the traditional methods.

2 Development of Exponential General Linear Methods

The exact representation of the solution of problem (1.1) is

$$y(t_n+h) = e^{hW}y(t_n) + e^{(t_h+h)W}\int_{t_n}^{t_n+h} e^{-\tau W}N(y,\tau)d\tau \tag{2.1}$$

Exponential time differencing scheme arises from approximating $N(y(\tau),\tau)$ by a polynomial $p(\theta)$ and then integrating exactly.

The aim of this paper is to develop a new approach to the construction and numerical analysis of extended Exponential General Linear Methods (EEGLM) for solving problems (1.1). [2] constructed the practical General Linear Methods with a considerable advantages over [3]. However, [5] extended the internal stages to the second level. This extension enables for the construction of methods of higher order. However, this present study is concerned with the extension of the numerical solution to a second level. This extension has not been seen anywhere in literature.

For given starting values $y_0, y_1, \cdots y_{q-1}$, the theoretical approximation y_{n+1} at time t_{n+1},$n \leq q-1$, is given by the recurrence relation or formula

$$y_{n+1} = e^{hW}y_n + h\sum_{i=1}^{s} B_i(hW)N(Y_{ni}) + h\sum_{k=1}^{q-1} V_k(hW)N(y_{n-k}) \tag{2.2}$$

The internal stages Y_{ni}, $1 \leq i \leq s$, are defined through

$$Y_{ni} = e^{c_i hW}y_n + h\sum_{j=1}^{i-1} A_{ij}(hW)(hW)N(y_{nj}) + h\sum_{k=1}^{q-1} U_{ik}(hW)N(y_{n-k}) \tag{2.3}$$

Our interest is to extend (2.2) by a higher exponential and its related matrix functions. The extended methods becomes

$$y_{n+1} = e^{hW}y_n + h\sum_{i=1}^{s} B_i(hW)N(Y_{ni}) + h\sum_{k=1}^{q-1} V_k^{(1)}(hW)N(y_{n-k}) + h^2\sum_{k=1}^{q-1} V_k^{(2)}(hW)N'(y_{n-k}) \tag{2.4}$$

The internal stages Y_{ni}, $1 \leq i \leq s$, are defined through

$$Y_{ni} = e^{c_i hW} y_n + h \sum_{j=1}^{i-1} A_{ij}(hW) N(Y_{nj}) + h \sum_{k=1}^{q-1} U_{ik}(hW) N(y_{n-k})$$

We assume throughout this paper that these conditions $U_{ik}(hL) = 0$ which implies $c_1 = 0$ and thus $y_{n1} = y_n$ are satisfied. The coefficients can be represented in a table 1

Before constructing methods arising from this met hod class, we derive the order conditions.

Table 1:Coefficient Tableau of EEGLM

A_{21} $\vdots \ddots$ $A_{s1} \cdots A_{s,s-1}$	$U_{21} \cdots U_{2,q-1}$ $\vdots \cdots \vdots$ $U_{s1} \cdots U_{s,q-1}$	
$B_1 \cdots B_{s-1} B_s$	$V_1^{(1)} \cdots V_{q-1}^{(1)}$	$V_1^{(2)} \cdots V_{q-1}^{(2)}$

3 Order Conditions for the Proposed Methods

In this section, we shall derivive the order conditions for the method (2.4). We assume that (1.1) is sufficiently regular. In particular, we require that the nonlinearity evaluated at the exact solution $f(t) = N(y(t))$ is sufficiently often differentiable with respect to t for $0 < t < T$. Before deriving the order conditions for this method, we state and prove the lemma below which will form the basis of proving the order conditions.

Lemma 1. The exact solution of the initial value problem

$$y' = Wy + f(t), t \geq t_n, y(t_n) \ \ given \tag{3.1}$$

has the following representation

$$y(t_n + \tau) = e^{W\tau} y(t_n) + \sum_{=0}^{m-1} \eta_{\ell+1}(W\tau) f^{(\ell)}(t_n) + R_m(m, \tau) \tag{3.2}$$

With

$$R_m(m, \tau) = \int_0^{\tau} r^{(\tau-\sigma)} \int_0^{\sigma} \frac{(\sigma - \xi)^{m-1}}{(m-1)!} f^{(m)}(t_m + \xi) d\xi d\sigma$$

Proof.

The exact representation of (3.5) is

$$y(t_n + \tau) = e^{\tau W} y(t_n) + \int_0^1 e^{(1-\sigma)} f(t_n + \sigma) d\sigma \tag{3.3}$$

Expanding $f(t_n + \sigma)$ via taylor series we have

$$f(t_n + \sigma) = f(t_n) + \sigma f(t_n) + \frac{\sigma^2}{2!} f^{(2)}(t_n) + \cdots +$$

$$\frac{\sigma^{m-1}}{(m-1)!} f^{(m-1)}(t_n + \int_0^{\sigma} \frac{(\sigma - \xi)^m}{(m!)} f^{(m)}(t_n + \xi) d\xi \tag{3.4}$$

Inserting equation (3.4) into equation (3.3) above, we have

$$y(t_n+\tau) = e^{\tau W}y(t_n) + \int_0^1 e^{(1-\sigma)}[f(t_n) + \sigma f^1(t_n) + \frac{\sigma^2}{2!}f^{(2)}f(t_n) + \cdots + \frac{\sigma^{m-1}}{(m-1)!}f^{m-1}f(t_n]d\sigma$$

$$+ \int_0^\tau \frac{(\sigma-\xi)^{m-1}}{(m-1)!}f^m f(t_n+\epsilon)d\sigma \tag{3.6}$$

Applying the definition of the η functions, which is defined as

$$\eta_\ell(z) = \frac{1}{\ell-1}! \int_0^1 e^{(1-\tau)z}\tau^{(\ell-1)}$$

we have

$$y(t_n+\tau) = e^{\tau W}y(t_n) + \sum_{\ell=0}^{m-1}\eta_{\ell+1}f^{(\ell)}(t_n) + R_n(m,\tau) \tag{3.7}$$

Where

$$R_n(m,\tau) = \int_0^\tau r^{(\tau-\sigma)}\int_0^\tau \frac{(\sigma-\xi)^{m-1}}{(m-1)!}f^{(m)}f(t_n+\xi)d\sigma, \tau \geq 0$$

Now back to our class of methods

$$y_{n+1} = e^{hW}y_n + h\sum_{i=1}^{s}B_i(hW)N(Y_{ni}) + h\sum_{k=1}^{q-1}V_k^{(1)}(hW)f(t_n-kh) + h^2\sum_{k=1}^{q-1}V_k^{(2)}(hW)f'(t_n-kh) \tag{3.8}$$

and the internal stages are defined through

$$Y_{ni} = e^{c_i hW}y_n + h\sum_{j=1}^{i-1}A_{ij}(hW)f(t_n+c_ih) + h\sum_{k=1}^{q-1}U_{ik}(hW)f(t_n-kh) \tag{3.9}$$

Expanding the functions in (3.8) and (3.9) above, we have

$$f(t_n+c_ih) = f(t_n) + c_ihf'(t_n) + \frac{(c_ih)^2}{2!}f''(t_n) + \frac{(c_ih)^3}{3!}f^{(3)}(t_n)$$

$$+ \cdots + \frac{(c_ih)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,m) \tag{3.10}$$

Where,

$$R(s,m) = \int_0^\tau \frac{(\sigma-\xi)^m}{m!}f^{(m)}(t_n+\xi)d\xi$$

$$f(t_n-kh) = f(t_n) - (kh)f'(t_n) + \frac{(-kh)^2}{2!}f''(t_n) + \frac{(-kh)^3}{3!}f^{(3)}(t_n)$$

$$+ \cdots + \frac{(-kh)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,r) \tag{3.11}$$

$$f'(t_n-kh) = f'(t_n) + (-kh)f''(t_n) + \frac{(-kh)^2}{2!}f^{(3)}(t_n) + \frac{(-kh)^3}{3!}f^{(iv)}(t_n)$$

$$+ \cdots + \frac{(-kh)^{m-1}}{(m-1)!}f^{(m)}(t_n) + R(s,m) \tag{3.12}$$

$$f(t_n+c_ih) = f(t_n) + c_ihf'(t_n) + \frac{(c_ih)^2}{2!}f''(t_n) + \frac{(c_ih)^3}{3!}f^{(3)}(t_n)$$

$$+ \cdots + \frac{(c_ih)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,m) \tag{3.13}$$

Substituting where appropriate into (3.8) and (3.9), we have

$$y_{n+1} = e^{hW}y_n + h\sum_{i=1}^{s} B_i N(Y_{ni}) + h\sum_{k=1}^{q-1} V_k^{(1)}(hW)[f(t_n) - (kh)f'(t_n) + \frac{(-kh)^2}{2!}f''(t_n) + \frac{(-kh)^3}{3!}f^{(3)}(t_n)$$
$$+ \cdots + \frac{(-kh)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,r)$$
$$+h^2\sum_{k=1}^{q-1} V_k^{(2)}(hW)[f'(t_n) + (-kh)f''(t_n) + \frac{(-kh)^2}{2!}f^{(3)}(t_n) + \frac{(-kh)^3}{3!}f^{(iv)}(t_n)$$
$$+ \cdots + \frac{(-kh)^{m-1}}{(m-1)!}f^{(m)}(t_n) + R(s,m)] \tag{3.14}$$

$$Y_{ni} = e^{c_i hW}y_n + h\sum_{j=1}^{i-1} A_{ij}[f(t_n) + c_j h f'(t_n) + \frac{(c_j h)^2}{2!}f''(t_n) + \frac{(c_j h)^3}{3!}f^{(3)}(t_n) +$$
$$\cdots + \frac{(c_j h)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,m)]$$
$$+h\sum_{k=1}^{q-1} U_{ik}[f(t_n) - (kh)f'(t_n) + \frac{(-kh)^2}{2!}f''(t_n) + \frac{(-kh)^3}{3!}f^{(3)}(t_n) +$$
$$\cdots + \frac{(-kh)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,r) \tag{3.15}$$

Substituting the exact solution values

$$y_{n1} = y_n, Y_{ni} = y(t_n + c_i h), 1 \le i \le s, n \ge 0 \tag{3.16}$$
$$y(t_n + c_i h) = e^{c_i hW}y_n + c_i h\int_0^1 e^{(1-\tau)c_i hW}f(t_n + c_i h)d\tau \tag{3.17}$$
$$f(t_n + c_i h) = f(t_n) + (hc_i)f'(t_n) + \frac{(hc_i)^2}{2!}f''(t_n) + \frac{(hc_i)^3}{3!}f'''(t_n) +$$
$$\cdots + \frac{(hc_i)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,m) \tag{3.18}$$

Inserting (3.18) into (3.17)

$$y(t_n + c_i h) = e^{c_i hL}y_n + c_i h\int_0^1 e^{(1-\tau)c_i hL}[f(t_n) + (hc_i)f'(t_n) + \frac{(hc_i)^2}{2!}f''(t_n)$$
$$+\frac{(hc_i)^3}{3!}f'''(t_n) + \cdots + \frac{(hc_i)^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + R(s,m)] \tag{3.19}$$

Similarly, the exact representation of the method solution is

$$y(t_n + h) = e^{hW}y_n + h\int_0^1 e^{(1-\tau)hW}f(t_n + c_i h)d\tau \tag{3.20}$$

But

$$f(t_n + h) = f(t_n) + hf'(t_n) + \frac{(h)^2}{2!}f''(t_n) + \frac{h^3}{3!}f'''(t_n) +$$
$$\cdots + \frac{h^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + G(s,m) \tag{3.21}$$

Substituting (3.21)into (3.20) we have

$$y(t_n+h) = e^{hW}y_n + h\int_0^1 e^{(1-\tau)hW}[f(t_n) + hf'(t_n) + \frac{h^2}{2!}f''(t_n) + \frac{h^3}{3!}f'''(t_n) + \cdots + \frac{h^{m-1}}{(m-1)!}f^{(m-1)}(t_n) + E(s,m)]d\tau \tag{3.22}$$

Subtracting (3.15) from (3.19) which denotes the defect (H_{ni}) of the internal stages and expressing them in terms of the power of h, we have

$$h^1 := c_i\int_0^1 e^{(1-\tau)c_ihW}f(t_n)d\tau - \sum_{j=1}^{i-1}A_{ij}f(t_n) - \sum_{k=1}^{q-1}U_{ik}f(t_n)$$

$$h^2 := \frac{c_i^2}{1!}\int_0^1 e^{(1-\tau)c_ihW}f'(t_n)d\tau - c_j\sum_{j=1}^{i-1}A_{ij}f'(t_n) - (-k)\sum_{k=1}^{q-1}U_{ik}f'(t_n)$$

$$h^3 := \frac{c_i^3}{2!}\int_0^1 e^{(1-\tau)c_ihW}f''(t_n)d\tau - \frac{c_j^2}{2!}\sum_{j=1}^{i-1}A_{ij}f''(t_n) - \frac{(-k)^2}{2!}\sum_{k=1}^{q-1}U_{ik}f''(t_n)$$

$$\vdots$$

$$h^m := \frac{c_i^m}{(m-1)!}\int_0^1 e^{(1-\tau)c_ihW}f^{(m-1)}(t_n)d\tau - \frac{c_j^{m-1}}{(m-1)!}\sum_{j=1}^{i-1}A_{ij}f^{(m-1)}(t_n) - \frac{(-k)^{m-1}}{(m-1)!}\sum_{k=1}^{q-1}U_{ik}f^{(m-1)}(t_n) \tag{3.23}$$

$$H_{ni} = hc_i^1\eta_1(c_ihW) + h^2c_i^2\eta_2(c_ihW) + h^3c_i^3\eta_3(c_ihW) + \cdots + h^Qc_i^Q\eta_Q(c_ihW) - \sum_{j=1}^{i-1}\frac{c_j^{\ell-1}}{(\ell-1)!}A_{ij}(hW)f^{(\ell-1)}(t_n) - \sum_{k=1}^{q-1}\frac{(-k)^{\ell-1}}{(\ell-1)!}U_{ik}(hW)f^{(\ell-1)}(t_n) \tag{3.24}$$

$$H_{ni} = \sum_{\ell-1}^{Q}h^\ell D_{\ell i}(hW)f^(\ell-1)(t_n) \tag{3.25}$$

where

$$D_{\ell i}(hW) = c_i^\ell\eta_\ell(c_ihW) - \sum_{j=1}^{i-1}\frac{c_j^{\ell-1}}{(\ell-1)!}A_{ij} - [\sum_{k=1}^{q-1}\frac{(-k)^{\ell-1}}{(\ell-1)!}U_{ik}(hW)] \tag{3.26}$$

Likewise, the numerical solution defects z_{n+1} equals

$$z_{n+1} = \sum_{\ell=1}^{p}h^\ell E_\ell(hW)f^{(\ell-1)}(t_n) + \cdots \tag{3.27}$$

where

$$E_\ell(hW) = \eta_\ell(hW) - \sum_{\ell=1}^{s}\frac{c_i^{\ell-1}}{(\ell-1)!}B_i(hW) - \sum_{k=1}^{q-1}\frac{(k-1)^{\ell-1}}{(\ell-1)!}V_k^{(1)}(hW) - \sum_{k=1}^{q-1}\frac{(k-1)^{\ell-2}}{(\ell-2)!}V_k^{(2)}(hW) \tag{3.28}$$

Remark 1. Our numerical scheme (3.3) is said to be of stage order Q and and general order P if $H_{ni} = 0(h^{Q+1})$ and $z_{n+1} = 0(h^{P+1})$. That is, requiring $D_{\ell i}(hW) = 0$ and $E_\ell(hW) = 0$. So, we obtain the order conditions

$$c_i^\ell\eta_\ell(c_ihW) = \sum_{j=1}^{i-1}\frac{c_j^{\ell-1}}{(\ell-1)!}A_{ij}(hW) + \sum_{k=1}^{q-1}\frac{(-k)^{\ell-1}}{(\ell-1)!}U_{ik}(hW) \tag{3.29}$$

and

$$\eta_\ell(hW) = \sum_{i=1}^{s} \frac{c_i^{\ell-1}}{(\ell-1)!} B_i(hW) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-1}}{(\ell-1)!} V_k^{(1)}(hW) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-2}}{(\ell-2)!} V_k^{(2)}(hW) \tag{3.30}$$

and so by definition $c_i = 0$ for all $1 \le i \le s$.

3.1 Construction of Various Extended Exponential General Linear Methods

3.1.1 Two-Stage Schemes

We discuss exponential general linear methods (2.4) with $p = 2, s = 2$ and $q = 2$ requiring the order and stage order conditions (3.29) and (3.30) to be fulfilled.
using the order conditions (3.29) and (3.30) to determine coefficients, we have

$$c_1^0 A_{21} + (-1)^0 U_{21} = \eta_1 \tag{3.31}$$

$$A_{21} + U_{21} = \eta_1$$

$$c_1^1 A_{21} - U_{21} = \eta_2 \tag{3.32}$$

$$U_{21} = -\eta_2$$

$$A_{21} = \eta_1 + \eta_2$$

Similarly,

$$c_1^1 B_1 + c_2^1 B_2 + (-1) V_1^{(1)} + (-1)^0 V_1^{(2)} = \eta_2$$

$$B_2 - V_1^{(1)} + V_1^{(2)} = \eta_2. \tag{3.33}$$

$$\frac{c_1^2 B_1}{2} + \frac{c_2^2 B_2}{2} + \frac{(-1)^2 V_1^{(1)}}{2} + (-1) V_1^{(2)} = \eta_3$$

$$\frac{B_2}{2} + \frac{V_1^{(1)}}{2} - V_1^{(2)} = \eta_3. \tag{3.34}$$

$$\frac{c_1^3 B_1}{3!} + \frac{c_2^3 B_2}{3!} + \frac{(-1) V_1^{(1)}}{3!} + \frac{(-1)^2 V_1^{(2)}}{2!} = \eta_4$$

$$\frac{B_2}{6} - \frac{V_1^{(1)}}{6} + \frac{V_1^{(2)}}{2} = \eta_4. \tag{3.35}$$

Solving equations (3.33) to (3.35) simultaneously, we have

$$B_2 = \frac{1}{4}\eta_2 + 4\eta_3 + 6\eta_4$$

$$V_1^{(1)} = \frac{5\eta_2}{4} + \eta_3 + \frac{9\eta_4}{2}$$

$$V_1^{(1)} = \frac{1}{2}[\eta_2 + 6\eta_4]$$

3.1.2. Extended Exponential General Linear Methods of Order Three Step Two Stage Order Two

The extended exponential general linear methods order three step two stage order two (known as methods 322)

Again, making use of the order conditions (3.29 and (3.30), we have

$$c_1^1 A_{21} + (-1)^0 U_{21} = \eta_1$$

$$A_{21} + A + U_{21} = \eta 1 \tag{3.36}$$

$$c_1^1 A_{21} + (-1) U_{21} = \eta_2$$

$$U_{21} = -\eta_2$$

$$A_{21} = \eta_1 + \eta_2$$

Similarly,

$$c_1^1 B_1 + c_2^1 B_2 + (-1) V_1^{(1)} + (-1)^0 V_1^{(2)} = \eta_2$$

$$B_2 - V_1^{(1)} + V_1^{(2)} = \eta_2 \tag{3.37}$$

$$\frac{c_1^2 B_1}{2} + \frac{c_2^2 B_2}{2} + \frac{(-1)^{(2)} V_1}{2} - V_1^{(2)} = \eta_3$$

$$\frac{B_2}{2} + \frac{V_1}{2} - V_1^{(2)} = \eta_3 \tag{3.38}$$

$$\frac{c_1^3 B_1}{3!} + \frac{c_2^3 B_2}{3!} + \frac{(-1)^{(3)} V_1}{3!} + \frac{(-1)^2 V_1^{(2)}}{2!} = \eta_4$$

$$\frac{B_2}{6} - \frac{V_1}{6} + \frac{V_1^{(2)}}{2} = \eta_4 \tag{3.39}$$

Solving (3.37)to (3.39) simultaneously, we have

$$B_2 = \frac{1}{4}[\eta_2 + 4\eta_3 + 6\eta_4]$$

$$V_1^{(1)} = \frac{-5\eta_2}{4} + \eta_3 + \frac{9\eta_4}{2}$$

$$V_1^{(2)} = \frac{1}{2}[6\eta_4 - \eta_2]$$

4 Stability Consideration of the Method

Stability is the property of a numerical method to keep the errors bounded as the computation advances. To investigate the stability criteria of our scheme, we adopt the root locus approach. This approach was used by [4] and [6]

4.1 Stabilities of Order Two Step Two and Stage Order One

We recall order two step two and stage order one (221) Scheme

$$y_{n+1} = e^{hL} y_n + hB_1(hL)N(Y_{n1}) + hB_2(hL)N(Y_{n2}) \tag{4.1}$$

With $Y_{n1} = y_n$

$$Y_{n2} = e^{c_2 hL} y_n + hA_{21}^{(2)}(hL)N(y_n) + h^2 A_{21}^{(2)} N'(y_n) \tag{4.2}$$

The first characteristic polynomial is given by

$$y_{n+1} = e^z y_n = 0 \; (hL = z)$$

Dividing through by y_n

$$\frac{y_{n+1}}{y_n} - \frac{e^z y_n}{y_n} = 0$$
$$r_n - e^z = 0 \tag{4.3}$$

The stability graph is shown i figure 1 below.

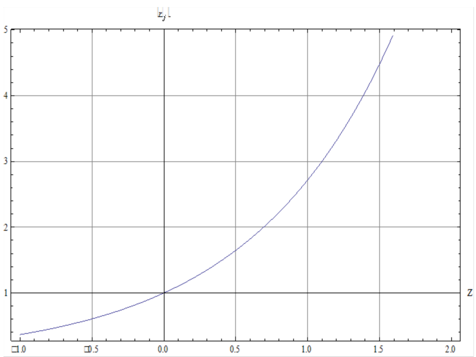

Figure 1:Stability of Order Two Step Two and Stage Order One (221) Scheme

The graph shows that the method is zero stable since the first characteristic polynomial lies in a unit circle.

4.2 Stabilities of Order Three Step Two and Stage Order Two

We recall order three step two and stage order two (322) Scheme given by

$$y_{n+1} = e^{hL} y_n + hB_1(hL)N(Y_{n1}) + hB_2(hL)N(Y_{n2}) + hV_1 N(y_{n-1}) \tag{4.4}$$

With $Y_{n1} = y_n$

$$Y_{n2} = e^{c_2 hL} y_n + hA_{21}^{(2)}(hL)N(y_n)$$
$$+hU_{21}^{(1)}(hL)N(y_{n-1}) + h^2 A_{21}^{(2)} N'(y_n) + h^{(2)} U_{21}^{(2)} N'(y_{n-1}) \tag{4.5}$$

Again the first characteristic polynomial is given by

$$y_{n+1} = e^z y_n = 0 \; (hL = z)$$

Dividing through by y_n

$$\frac{y_{n+1}}{y_n} - \frac{e^z y_n}{y_n} = 0$$
$$r_n - e^z = 0 \tag{4.6}$$

The stability graph of order three step two and stage order two (322) Scheme is shown in figure 2 below

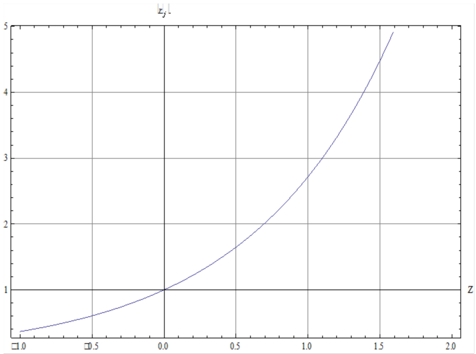

Figure 2:The Stability of Order Three Step Two and Stage Order Two (322)

The graph shows by MATLAB that order three step two and stage order two (322) Scheme is zero stable as the first characteristics polynomial lies in a unit circle

5 Numerical Experiments

In this section, we discuss by comparing the accuracies of exponential general linear methods in recent literatures with the different extended exponential general linear methods constructed in this paper.

Problem 1. Given $y' = 2y,\ y(0) = 3$. The theoretical solution is given as $y(t) = 3exp(2t)$

The accumulated errors of EEGLM 322,EEGLM 221, 322,Butcher and Wright and Calvo and Palencia for the above problem with their corresponding meshsizes are shown in the figure 3 below.

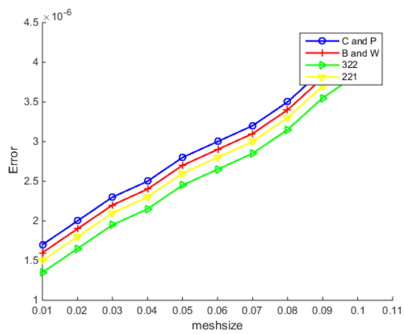

Figure 3:The graph of the accuracies of EEGLM 322, EEGLM 221, C and P, W and B

Clearly,our scheme indicates considerable improvement over Butcher and Wight and Calvo and Palencia.

Problem 2. Given $y' = -10(y-1)^2,\ y(0) = 2,$ with $t \in [0, 1]$

The theoretical solution is given as $y = \frac{2+10t}{1+10t}$

The accumulated errors of EEGLM 322,EEGLM 221,Butcher and Wright and Calvo and Palencia for the above problem with their corresponding meshsizes are shown in the figure 4 below

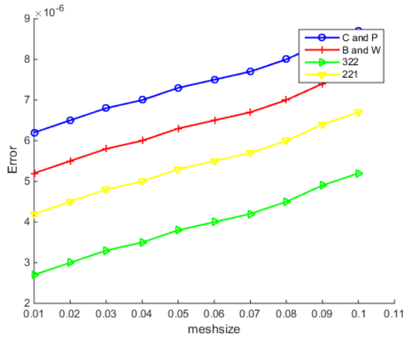

Figure 4:The graph of the accuracies of EEGLM 322, EEGLM 221, C and P, W and B

Clearly,our scheme indicates considerable improvement over Butcher and Wight and Calvo and Palencia

6 Conclusions

The numerical results obtained through our proposed schemes as indicated in figure 3 and 4, exhibit a considerable improvement over the existing methods in [2] and [3]. The numerical results presented also show that our scheme is accurate and efficient in handling the given IVP. Finally, the stability analysis shows that the schemes are stable as the parasitic root lie in the unit disk.

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