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Functional Difference Equations

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ABSTRACT. The purpose of this paper is to generalize some results of Horadam and to extend some identities due to Carlitz in the context of functional difference equations. It includes an operator which yields a type of Appell-set criterion. Generalizations of Simson's identity occur as by-products and these are linked to some recent advances in the literature.

1 Introduction

The purpose of this paper it to prove two results to generalize aspects of Horadam's generalized sequence of numbers, $\{w_n(a,b;p,q)\}$, defined by the second order linear homogeneous recurrence relation

$$w_n = pw_{n-1} - qw_{n-2}, \quad n > 2. \tag{1}$$

with initial conditions $w_0 = a, w_1 = b$[1] with fundamental properties [2,3,4]. Carlitz had also previously looked at some of the arithmetic properties of generalized sequences [5] and functional difference equations [6], and we build on some of his ideas here too. Table 1 illustrates the efficiency and effectiveness of Horadam's notation in placing these sequences in perspective for comparisons.

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a	b	p	q	w_n	*Sequence*
0	1	1	-1	F_n	Fibonacci
2	1	1	-1	L_n	Lucas
0	1	2	-1	P_n	Pell
1	3	2	-1	Q_n	Pell-Lucas
0	1	x+2	+1	$B_n(x)$	Morgan-Voyce Even Fibonacci
1	1	x+2	+1	$b_n(x)$	Morgan-Voyce Odd Fibonacci
2	x+2	x+2	+1	$C_n(x)$	Morgan-Voyce Even Lucas
-1	1	x+2	+1	$c_n(x)$	Morgan-Voyce Odd Lucas
0	1	1	-x	$J_n(x)$	Jacobsthal-Fibonacci
2	1	1	-x	$j_n(x)$	Jacobsthal-Lucas
0	1	x	+1	$V_n(x)$	Vieta-Fibonacci
2	x	x	+1	$v_n(x)$	Vieta-Lucas
a	a+d	2	+1	A_n	Arithmetic
a	ar	r	0	G_n	Geometric

More recent further generalizations relevant to the subsequent development in this paper include coupled and multiplicative Fibonacci sequences [7,8] and particularly the table in the second scheme [9: 331].

2 Notation

We define,

$$w_n(x) = \sum_{k=0}^{\infty} w_{n+k} \frac{x^k}{k!}, \text{where } n = 0, 1, 2, .. \tag{2}$$

so that we get the differential equation

$$\frac{d}{dx} w_n(x) = w_{n+1}(x), \tag{3}$$

and the difference equation

$$w_n(x) = p w_{n-1}(x) - q w_{n-2}(x), \text{where } n > 2. \tag{4}$$

This is slightly more general than that studied by Carlitz [7], which is itself a generalization of the results of Elmore [8]. The general term of $\{w_n(a, b; p, q)\}$ is well known:

$$w_n = A\alpha^n + B\beta^n \tag{5}$$

where α, β are the zeroes of $x^2 - px + q$ and $p = \alpha + \beta,\ q = \alpha\beta,\ d = \alpha - \beta,\ E = pab - qa^2 - b^2$ and from the initial conditions

$$A = \frac{b - a\beta}{d},\ B = \frac{a\alpha - b}{d}$$

so that

$$E = ABd^2$$

Now, from (5) we define,

$$\begin{aligned} w_n(x) &= \sum_{k=0}^{n} (A\alpha^{n+k} + B\beta^{n+k}) \frac{x^k}{k!} \\ &= A\alpha^n e^{ax} + B\beta^n e^{bx} \end{aligned}$$

It follows immediately that

$$\begin{aligned}\sum_{k=0}^{\infty} w_{n+k}(x)\frac{y^k}{k!} &= \sum_{k=0}^{\infty}(A\alpha^{n+k}e^{\alpha x} + B\beta^{n+k}e^{\beta x})\frac{y^k}{k!}\\ &= A\alpha^n e^{\alpha(x+y)} + B\beta^n e^{\beta(x+y)}\\ &= w_n(x+y)\end{aligned}$$

Horadam [2] observed that for $n = 0$,

$$\begin{aligned}w(x) &= w_0(x)\\ &= \frac{a+(b+pa)x}{1-px+qx^2}\end{aligned}$$

This is a general case of what Long and Crawford observed as a "surprising new insight into the fact that the sum of the diagonal elements in Pascal's triangle are the Fibonacci numbers" [9], namely [10]:

$$\frac{1}{89} = \frac{11^0}{10^2} + \frac{11^1}{10^4} + \frac{11^2}{10^6} + \frac{11^3}{10^8} + \ldots$$

This can be extended to

$$\frac{a(s-p)+b}{s^2-ps+q} = \frac{a}{s} + \frac{b}{s^2} + \frac{pb-qa}{s^3} + \ldots$$

$s^2 - ps + q > 0$, which can be numerically exemplified with $s = 10$. Long et al [11] later also looked at some of the other related arithmetic properties of these generalized sequences. Theorem 7 of Falcon and Plaza [15] leads to an elegant generalization of this section.

3 Generalization of Carlitz

Consider next,

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(w_{m+r}(u)w_{n+r}(v) - w_r(x)w_{m+n+r}(y))\frac{s^m t^n}{m!n!}\\ &= w_r(u+s)w_r(v+t) - w_r(x)w_r(y+s+t)\\ &= (A\alpha^r e^{\alpha(u+s)} + B\beta^r e^{\beta(u+s)})(A\alpha^r e^{\alpha(v+t)} + B\beta^r e^{\beta(v+t)})\\ &\quad - (A\alpha^r e^{\alpha x} + B\beta^r e^{\beta x})(A\alpha^r e^{\alpha(y+s+t)} + B\beta^r e^{\beta(y+s+t)})\\ &= A^2\alpha^{2r}e^{\alpha(u+v+s+t)} + B^2\beta^{2r}e^{\beta(u+v+s+t)} + Ed^{-2}q^r(e^{\alpha(u+s)+\beta(v+t)} + e^{\beta(u+s)+\alpha(v+t)})\\ &\quad - A^2\alpha^{2r}e^{\alpha(x+y+s+t)} + B^2\beta^{2r}e^{\beta(x+y+s+t)} + Ed^{-2}q^r(e^{\alpha x+\beta(y+s+t)} + e^{\beta x+\alpha(y+s+t)})\end{aligned}$$

If we take $x + y = s + t$, this reduces to

$$\begin{aligned}&Ed^{-2}q^r(e^{\alpha(u+s)+\beta(v+t)} + e^{\beta(u+s)+\alpha(v+t)} - e^{\alpha x+\beta(y+s+t)} - e^{\beta x+\alpha(y+s+t)})\\ &= Ed^{-2}q^r e^{px}(e^{\alpha(-x+u+s)+\beta(-x+v+t)} + e^{\alpha(-x+v+t)+\beta(-x+u+s)} - e^{\beta(-x+y+s+t)} - e^{\alpha(-x+y+s+t)})\\ &= Ed^{-2}q^r e^{px}((-e^{\alpha(-x+v+t)} + e^{\beta(-x+v+t)})(e^{\alpha(-x+u+s)} - e^{\beta(-x+u+s)}))\\ &= -Eq^r e^{px}\mathcal{U}_0(-x+u+t)\mathcal{U}_0(-x+v+t)\end{aligned}$$

in which

$$\mathcal{U}_n(x) = \sum_{k=0}^{\infty} \mathcal{U}_{n+k} \frac{x^k}{k!}$$

and

$$\mathcal{U}_n = u_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{6}$$

for $\{w_n(1, p; p, q)\}$, Lucas' fundamental sequence [12]; and so, when $r = 2$:

$$\mathcal{U}_n(x) = d^{-1}(\alpha^n e^{\alpha x} - \beta^n e^{\beta x})$$

which agrees with Barakat [13]. That is, we have proved that when $x + y = u + v$,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (w_r(x) w_{m+n+r}(y) - w_{m+r}(u) w_{n+r}(v)) \frac{s^m t^n}{m! n!} = -Eq^r e^{px} \mathcal{U}_0(u + s - x) \mathcal{U}_0(v + t - x) \tag{7}$$

But since, from above,

$$\sum_{k=0}^{\infty} \mathcal{U}_{n+k}(x) \frac{y^k}{k!} = \mathcal{U}_n(x + y)$$

then

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{U}_m(-x + u) \mathcal{U}_n(-x + v) \frac{s^m t^n}{m! n!} = \mathcal{U}_0(u + s - x) \mathcal{U}_0(v + t - x)$$

and on equating coefficients of $s^m t^n$, we get,

$$w_r(x) w_{m+n+r}(y) - w_{m+r}(u) w_{n+r}(v) = Eq^r e^{px} \mathcal{U}_m(u - x) \mathcal{U}_n(v - x)$$

When $a = 0, b = 1, w_n(x) = \mathcal{U}_n(x)$, and we obtain

$$\mathcal{U}_{m+r}(u) \mathcal{U}_{n+r}(v) - \mathcal{U}_r(x) \mathcal{U}_{m+n+r}(y) = q^r e^{px} \mathcal{U}_m(u - x) \mathcal{U}_n(v - x) \tag{8}$$

which is essentially the same as Equation (2.3) of Carlitz [9]. Equation (6) is effectively treated elsewhere in the context of some general k-Fibonacci sequences [15,17].

4 Generalization of Horadam

If we replace m by $-r$, the original r by n, the original n by $r + t$ in (7), we then find that

$$\mathcal{U}_{n-r}(u) \mathcal{U}_{n+r+t}(v) - \mathcal{U}_n(x) \mathcal{U}_{n+t}(y) = q^n e^{px} \mathcal{U}_{-r}(u - x) \mathcal{U}_{r+t}(v - x) \tag{9}$$

which is directly comparable with Equation (4.18) of [1], in which Horadam defines negative subscripts. Horadam's Equation (4.18) has the form

$$w_{n-r} w_{n+r+t} - w_n w_{n+t} = q^{n-r} E u_{r-1} u_{r+t-1} \tag{10}$$

in which $u_n = \mathcal{U}_{n+1}$. If we contrast (9) and (10) we see that E in the latter performs a role akin to e in the former, which makes Horadam's choice of 'E' as symbol particularly apt. The analogy between e and E if we put $t = 0$ and $x = y = u = v$ in (7) and let as in [9] to obtain

$$\mathcal{U}_{n-r}(x) \mathcal{U}_{n+r}(x) - \mathcal{U}_n^2(x) = q^n e^{px} \mathcal{U}_{-r} \mathcal{U}_r \tag{11}$$

since $\mathcal{U}_n(0) = \mathcal{U}_n$. This is a generalization of Simson's identity [14] and is comparable with Equation (4.5) of Horadam [1], namely,

$$w_{n+r}w_{n-r} - w_n^2 = Eq^{n-r}u_{r-1}^2 \tag{12}$$

Other more recent and pertinent generalizations of Simson's identity may be found in [15] and [17; Theorems 9,10,11].

5 Further Extensions

We can also define

$$w_n^*(x) = w_n^*(x,\lambda) = \sum_{k=0}^{\infty} w_{n+k} \binom{x}{k} \lambda^k \tag{13}$$

Then $w_n^*(0) = w_n$, and

$$\begin{aligned} w_{n+1}^* &= \sum_{k=0}^{\infty} w_{n+k} \binom{x}{k} \lambda^k \\ &= \sum_{k=0}^{\infty} pw_{n+k} - qw_{n+k-1} \binom{x}{k} \lambda^k \\ &= pw_n^*(x) - qw_{n-1}^*(x) \end{aligned} \tag{14}$$

which differs slightly from the corresponding equation for h_{n+1}^* in [6]. Moreover, from (13) we define

$$\begin{aligned} \Delta_x w_n^*(x) &= w_n^*(x+1) - w_n^*(x) \\ &= \sum_{k=0}^{\infty} w_{n+k} \left(\binom{x+1}{k} - \binom{x}{k} \right) \lambda^k \\ &= \lambda \sum_{k=0}^{\infty} w_{n+k} \binom{x}{k-1} \lambda^{k-1} \\ &= \lambda \sum_{k=0}^{\infty} w_{n+k+1} \binom{x}{k} \lambda^k \end{aligned} \tag{15}$$

so that

$$\Delta_x w_n^*(x,\lambda) = \lambda w_{n+1}^*(x,\lambda)$$

which is not unlike an Appell set criterion [15]. Other recent analogues of extensions of Appell set criteria include Ernst [23] and Sadjang [24]. The latter expresses q-Appell sets in terms of the Hahn derivative and Fermatian numbers [25]. These in turn yield inequalities among products of Fibonacci numbers and their generalizations [26].

It shows that the power series in the definition of $w_n^*(x)$ converges for sufficiently small λ. From this definition and the general expression for w_n, we obtain

$$\begin{aligned} w_n^*(x) &= \sum_{k=0}^{\infty} \left(A\alpha^n \binom{x}{k} (\alpha\lambda)^n + B\beta^n \binom{x}{k} (\beta\lambda)^n \right) \\ &= A\alpha^n (1+\lambda\alpha)^x + B\beta^n (1+\lambda\beta)^x. \end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned} w_n^*(x+y) &= A\alpha^n (1+\lambda\alpha)^{x+y} + B\beta^n (1+\lambda\beta)^{x+y} \\ &= \sum_{k=0}^{\infty} \left[A\alpha^n (1+\lambda\alpha)^x + B\beta^n (1+\lambda\beta)^x \right] \binom{y}{k} \lambda^k \\ &= \sum_{k=0}^{\infty} w_{n+k}^*(x) \binom{y}{k} \lambda^k \end{aligned} \tag{17}$$

which is similar to, but different from, the comparable Equation (3.3) of [6]. Following Carlitz we write

$$\begin{aligned} &-q w_{m-1}^*(x) w_n^*(x) + w_m^*(x) w_{n+1}^*(y) \\ &\quad = -q(A\alpha^{m-1}(1+\lambda\alpha)^x + B\beta^{m-1}(1+\lambda\beta)^x) \times (A\alpha^n(1+\lambda\alpha)^y + B\beta^n(1+\lambda\beta)^y) \\ &\qquad + (A\alpha^m(1+\lambda\alpha)^x + B\beta^m(1+\lambda\beta)^x) \times (A\alpha^{n+1}(1+\lambda\alpha)^y + B\beta^{n+1}(1+\lambda\beta)^y) \\ &\quad = A^2(-q\alpha^{m+n-1} + \alpha^{m+n+1})(1+\lambda\alpha)^{x+y} + B^2(-q\beta^{m+n-1} + \beta^{m+n+1})(1+\lambda\beta)^{x+y} \\ &\qquad + Ed^{-2}((-q\alpha^{m-1}\beta^n + \alpha^m\beta^{n+1})(1+\lambda\alpha)^x(1+\lambda\beta)^y \\ &\qquad + (-q\alpha^n\beta^{m-1} + \alpha^{n+1}\beta^m)(1+\lambda\alpha)^y(1+\lambda\beta)^x). \end{aligned} \tag{18}$$

Since $\alpha\beta = q$ and $\alpha^2 - q = \alpha(\alpha-\beta)$, $\beta^2 - q = -\beta(\alpha-\beta)$, this reduces to

$$\begin{aligned} &A^2(\alpha^{m+n-1}(\alpha^2-q))(1+\lambda\alpha)^{s+t} + B^2(\beta^{m+n-1}(\beta^2-q))(1+\lambda\beta)^{s+t} \\ &\quad = A^2(\alpha-\beta)\alpha^{m+n}(1+\lambda\alpha)^{s+t} - B^2(\alpha-\beta)\beta^{m+n}(1+\lambda\beta)^{s+t} \\ &\quad = A(b-\alpha\beta)\alpha^{m+n}(1+\lambda\alpha)^{s+t} + B(b-\alpha\beta)\beta^{m+n}(1+\lambda\beta)^{s+t} \\ &\quad = b(A\alpha^{m+n}(1+\lambda\alpha)^{s+t} + B\beta^{m+n}(1+\lambda\beta)^{s+t}) - aq(A\alpha^{m+n-1}(1+\lambda\alpha)^{s+t} \\ &\qquad + B\beta^{m+n-1}(1+\lambda\beta)^{s+t}) \\ &\quad = b w_{m+n}^*(s+t) - aq w_{m+n-1}^*(s+t). \end{aligned} \tag{19}$$

We have therefore proved that

$$b w_{m+n}^*(s+t) - aq w_{m-n-1}^*(s+t) = w_m^*(s) w_{n+1}^*(t) - q w_{m-1}^*(s) w_n^*(t) \tag{20}$$

If we replace n by $n-1$ and use (14), we obtain

$$a w_{m+n}^*(s+t) - (b-pa) q w_{m+n-1}^*(s) = w_m^*(s) w_n^*(t) - q w_{m-1}^*(s) w_{n-1}^*(t) \tag{21}$$

which is a generalization of (4.1) of Horadam [1], since if $s = t = 0$ we have $w_m^*(0) = w_m$ and (21) becomes the known

$$a w_{m+n} + (b-pa) w_{m+n-1} = w_m w_n - q w_{m-1} w_{n-1}$$

The result (21) is more general then Equation (3.4) of [6] in which there are misprints in the third and fifth lines.

6 Concluding comments

In a similar way we can obtain another generalization of Simson's identity

$$\begin{aligned}
& w_{m-1}^*(s)w_{m+1}^*(s) - w_m^{*^2}(s) \\
&= (A\alpha^{m-1}(1+\lambda\alpha)^s + B\beta^{m-1}(1+\lambda\beta)^s)(A\alpha^{m+1}(1+\lambda\alpha)^s + B\beta^{m+1}(1+\lambda\beta)^s) \\
&\quad - (A\alpha^m(1+\lambda\alpha)^s + B\beta^m(1+\lambda\beta)^s)^2 \\
&= Ed^{-2}(\alpha^{m-1}\beta^{m+1} - 2\alpha^m\beta^m + \alpha^{m+1}\beta^{m-1})((1+\lambda\alpha)(1+\lambda\beta))^s \\
&= q^{m-1}Ed^{-2}(\beta^2 - 2\alpha\beta + \alpha^2)(1+p\lambda+q\lambda^2)^s \\
&= q^{m-1}EE_w^s
\end{aligned} \tag{22}$$

in which $E_w = 1 + p\lambda + q\lambda^2$. Thus we have,

$$q^m EE_w^x = w_m^*(x)w_{m+2}^*(x) - w_{m+1}^{*^2}(x) \tag{23}$$

as an extension of Horadam's generalization of Simson's identity which was expressed as

$$q^n E = w_n w_{n+2} - w_{n+1}^2.$$

Note that Ew is analogous to Horadam's E. Finally, we then consider

$$\begin{aligned}
& w_{n-r}^*(x)w_{n+r+t}^*(x) - w_n^*(x)w_{n+t}^*(x) \\
&= (A\alpha^{n-r}(1+\lambda\alpha)^x + B\beta^{n-r}(1+\lambda\beta)^x)(A\alpha^{n+r+t}(1+\lambda\alpha)^x + B\beta^{n+r+t}(1+\lambda\beta)^x) \\
&\quad - (A\alpha^n(1+\lambda\alpha)^x + B\beta^n(1+\lambda\beta)^x)(A\alpha^{n+t}(1+\lambda\alpha)^x + B\beta^{n+t}(1+\lambda\beta)^x) \\
&= Ed^{-2}(\alpha^{n-r}\beta^{n+r+t} + \alpha^{n+r+t}\beta^{n-r} - \alpha^n\beta^{n+t} - \alpha^{n+t}\beta^n)E_w^x \\
&= Ed^{-2}q^{n-r}(\beta^{2r+t} + \alpha^{2r+t} - \alpha^r\beta^{r+t} - \alpha^{r+t}\beta^r)E_w^x \\
&= Eq^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha-\beta}\right)\left(\frac{\alpha^{r+t} - \beta^{r+t}}{\alpha-\beta}\right)E_w^x \\
&= q^{n-r}Eu_{r-1}u_{r+t-1}E_w^x
\end{aligned} \tag{24}$$

for $\{u_n\}$ defined in (6). Thus, we have

$$w_{n-r}^*(x)w_{n+r+t}^*(x) - w_n^*(x)w_{n+t}^*(x) = q^{n-r}Eu_{r-1}u_{r+t-1}E_w^x \tag{25}$$

as a generalization of (4.18) of Horadam [1]. Further related research can connect these k-Fibonacci generalizations [27,28] to modular tings and integer structure analysis [29].

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