On the Class of the Rational Difference Equations

Osama Moaaz¹, Mahmoud A.E. Abdelrahman²
¹,²Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt.
Email:o_moaaz@mans.edu.eg, mahmoud.abdelrahman@mans.edu.eg

ABSTRACT. In this paper we investigate the qualitative behavior of the solution of the class of the rational Difference Equations. Namely, we consider the stability, boundedness, and periodicity of the solution. We also give some interesting examples in order to verify our results.

1 Introduction

Difference Equations prescribe real life situations in probability theory, genetics in biology, queuing theory, electrical network, psychology, physics, statistical problems, sociology, combinatorial analysis, economics, etc. So our study of the Difference Equations is so interesting. Therefore investigating new techniques to solve more complicated problems. There has been many work about the global asymptotic of solutions of rational difference equations, [1] - [17] and references therein. It is so important to investigate the asymptotic behavior of solutions of a class of nonlinear difference equations and to discuss the boundedness, periodicity and stability of their equilibrium points.

In this paper, we consider the analytical investigation of the solution of the following recursive sequence

\[ x_{n+1} = \frac{b x_{n-k}}{a + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} x_{n-i}} \]  

(1)

*Osama Moaaz
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where the initial conditions \(x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0\) are arbitrary positive real numbers and \(a, b, \alpha, \beta_j, j = 0, 1, \ldots k\) are positive constants.

In this section we present the basic definitions and theorems of the our model, namely equilibrium points, local and global stability, boundedness, periodicity and the oscillation of the solution.

**Definition 1.1 [13]** (Equilibrium point)

Consider a difference equation in the form

\[
x_{n+1} = F(x_{n-l}, x_{n-k}, x_{n-s}), \quad n = 0, 1, 2, \ldots
\]

where \(F\) is a continuous function, while \(l, k\) and \(s\) are positive integers. A point \(x\) is said to be an equilibrium point of the equation (2) if it is a fixed point of \(F\), i.e., \(x = F(x, x, x)\).

**Definition 1.2 [13]** (stability)

Let \(x \in (0, \infty)\) be an equilibrium point of equation (2). Then we have

(a) **local stability**

An equilibrium point \(x\) of equation (2) is said to be locally stable if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that, if \(x_{-\nu} \in (0, \infty)\) for \(\nu = 0, 1, \ldots, r\) with \(r \sum_{i=0}^{r} |x_{-i} - x| < \delta\), then \(|x_n - x| < \epsilon\) for all \(n \geq -r\).

(b) **local asymptotic stability**

An equilibrium point \(x\) of equation (2) is said to be locally asymptotically stable if it is locally stable and there exists \(\gamma > 0\) such that, if \(x_{-\nu} \in (0, \infty)\) for \(\nu = 0, 1, \ldots, r\) with \(r \sum_{i=0}^{r} |x_{-i} - x| < \gamma\), then

\[
\lim_{n \to \infty} x_n = x.
\]

(c) **global stability**

An equilibrium point \(x\) of equation (2) is said to be a global attractor if for every \(x_{-\nu} \in (0, \infty)\) for \(\nu = 0, 1, \ldots, r\) we have

\[
\lim_{n \to \infty} x_n = x.
\]

(d) **global asymptotic stability**

An equilibrium point \(x\) of equation (2) is said to be globally asymptotically stable if it is locally stable and a global attractor.
(v) **Unstability**

An equilibrium point $\mathbf{x}$ of equation (2) is said to be unstable if it is not locally stable.

**Definition 1.3** [13] **(Periodicity)**

A sequence $\{x_n\}_{n=-\infty}^{\infty}$ is said to be periodic with period $t$ if $x_{n+t} = x_n$ for all $n \geq -r$. A sequence $\{x_n\}_{n=-\infty}^{\infty}$ is said to be periodic with prime period $t$ if $t$ is the smallest positive integer having this property.

**Definition 1.4** [13] **(Boundedness)**

Equation (2) is called permanent and bounded if there exists numbers $m$ and $M$ with $0 < m < M < \infty$ such that for any initial conditions $x_{-n} \in (0, \infty)$ for $n = 0, 1, \ldots, r$ there exists a positive integer $N$ which depends on these initial conditions such that $0 < m < M < \infty$ for all $n \geq N$.

**Definition 1.5** [13] The linearized equation of equation (2) about the equilibrium point $\mathbf{x}$ is defined by the linear difference equation

$$z_{n+1} = \sum_{i=0}^{k} h_i z_{n-i}, \quad (3)$$

where

$$h_i = \frac{\partial F(x, x, \ldots, x)}{\partial x_{n-i}}, \quad k \text{ is positive integer}$$

**Theorem 1.1** [12] Assume that $h_i, i = 0, 1, \ldots, k \in \mathbb{R}$. Then

$$\sum_{i=0}^{k} |h_i| < 1,$$

is a sufficient condition for the asymptotic stability of equation (2).

The paper is structured as follows: In Section 2 we study the stability behaviour of the solution for equation (1) and give an interesting example to support our analysis. In Section 3, we prove that the positive solution of equation (1) is bounded. In Section 4 we study the periodic behaviour of the solution for the equation (1). We also give two examples to show how our model is so rich.

## 2 The stability of solutions

In this section we study the local stability character of the solutions of equation (1). The positive equilibrium points of equation (1) are given by

$$\mathbf{x} = 0, \quad \mathbf{x} = \left[ \frac{b - \alpha}{B} \right]^i, \quad \text{where } B = \sum_{j=0}^{k} \beta_j.$$
Now, we define the continuous function \( f : (0, \infty)^3 \rightarrow (0, \infty) \), such that

\[
f(u_0, u_1, ..., u_k) = \frac{bu_k}{\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} u_i},
\]

Therefore, it follows that

\[
\frac{\partial f}{\partial u_m} = -bu_k \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} u_i \quad \alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} u_i, \quad m = 0, 1, 2, ..., k-1.
\]  

(4)

and

\[
\frac{\partial f}{\partial u_k} = \frac{b(\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} u_i) - bu_k \sum_{j=0}^{k-1} \beta_j \prod_{i=0, i \neq j}^{k-1} u_i}{\left[\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} u_i\right]^2} = \frac{ba + b\beta_k \prod_{i=0}^{k-1} u_i}{\left[\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} u_i\right]^2}
\]

(5)

(6)

**Theorem 2.1** If \( \frac{b}{\alpha} < 1 \), then the equilibrium point \( \bar{x} = 0 \) of equation (1) is locally stable.

Proof: The linearized equation of (1) about the equilibrium point \( \bar{x} = 0 \) is the linear difference equation

\[
z_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, ..., \bar{x})}{\partial u_i} z_{n-i}.
\]

From equations (4), (5) we have

\[
\frac{\partial f}{\partial u_m}(\bar{x}, \bar{x}, ..., \bar{x}) = 0,
\]

for \( m = 0, 1, ..., k-1 \) and

\[
\frac{\partial f}{\partial u_k}(\bar{x}, \bar{x}, ..., \bar{x}) = \frac{b}{\alpha}.
\]

It is follows by [13, Theorem 1] that, equation (1) is locally stable at \( \bar{x} = 0 \) if

\[
\frac{b}{\alpha} < 1.
\]

Hence, the proof is completed.

The following example shows the stability of solution of equation (1) at \( \bar{x} = 0 \).

**Example 2.1** We consider the following initial data: \( x_{-2} = 1.01, x_{-1} = 0.99, x_0 \) for equation (1) with \( k = 1 \), Figure 1.
Theorem 2.2 If
\[ |T| ((k - 1)B + \beta_k) + |T\beta_k + \alpha B| < bB. \]
where \( T = (b - \alpha) \), then the equilibrium point \( \bar{X} \) of equation (1) is locally stable.

Proof: From equations (4), (5) we have
\[
\frac{\partial f}{\partial u_m}(\bar{X}, \bar{X}, \ldots, \bar{X}) = -\frac{b - \alpha}{bB} B^{[m]} = -h_m,
\]
for \( m = 0, 1, \ldots, k - 1 \), where \( B^{[m]} = B - \beta_m \) and
\[
\frac{\partial f}{\partial u_k}(\bar{X}, \bar{X}, \ldots, \bar{X}) = \frac{b\beta_k + \alpha B^k}{bB} = -h_k.
\]

Then the linearized equation
\[
z_{n+1} + \sum_{i=0}^{k} h_i z_{n-i} = 0 \quad (7)
\]
It follows by [13, Theorem 1] that, equation (1) is locally stable at \( \bar{X} = \left[ \frac{b - \alpha}{bB} \right]^2 \) if
\[
\sum_{i=0}^{k} |h_i| < 1.
\]
This implies that
\[
\left| \frac{b - \alpha}{bB} \left( \sum_{m=0}^{k-1} B^{[m]} \right) \right| + \left| \frac{b\beta_k + \alpha B^k}{bB} \right| < 1.
\]
and so,

$$|T| ((k-1)B + \beta_k) + |T\beta_k + aB| < bB.$$ 

Hence, the proof is completed.

The following example shows the stability of solution of equation (1) at $\overline{x} = \left[ \frac{b-a}{b} \right]^\frac{1}{k}$.

**Example 2.2** We consider the following initial data: $x_{-2} = 1.1, x_{-1} = 0.99, x_0 = 1.01$ for equation (1) with $k = 2$, Figure 2

3 Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of equation (1).

**Theorem 3.1** If $\frac{k}{k-1} \leq 1$ then the solutions of equation (1) are bounded.

Proof: Assume that $\{x_n\}_{n=-k}^\infty$ be a solution of equation (1). Then we have
\[
x_{n+1} = \frac{b x_{n-k}}{a + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} x_{n-i}},
\]

Hence, we have \( x_{n+1} \leq x_{n-k} \). Thus we can divided the sequence \( \{x_n\}_{n=-k}^{\infty} \) to \( k+1 \) subsequence bounded above by the initial conditions as follows:

\[
\begin{align*}
x_{-k} & \geq x_1 \geq x_{k+2} \geq x_{2k+3} \geq ... \\
x_{-k+1} & \geq x_2 \geq x_{k+3} \geq x_{2k+4} \geq ... \\
x_{-k+2} & \geq x_3 \geq x_{k+4} \geq x_{2k+5} \geq ... \\
& \vdots \\
x_0 & \geq x_{k+1} \geq x_{2k+2} \geq x_{3k+3} \geq ...
\end{align*}
\]

Hence we chose \( M = \max\{x_{-k}, x_{-k+1}, ..., x_0\} \), which leads to \( 0 \leq x_n \leq M \). Thus, the proof is completed.

\section{Periodic solutions}

In this section we give the periodic behaviour of the solution for the non linear difference equation (1). Moreover we give the periodic character of solutions of these equations of order \( k+1 \), which is not familiar.

\textbf{Theorem 4.1} Assume that \( \beta_j = 1 \), for \( j = 0, 1, 2, ..., k \) and at least one of the initial conditions \( x_{-k}, x_{-k+1}, ..., x_0 \neq 0 \). The equation (1) has period \( k+1 \) solution if

\[
b = (a + A),
\]

where \( A = \sum_{j=0}^{k} \prod_{i=0, i \neq j}^{k} x_{-i} \).

Proof: Suppose that there exists a distinct prime period \( k+1 \) solutions of equation (1). Thus, we have the following algebraic system of \( k+1 \) equations

\[
x_1 = x_{-k}, x_{-k+1} = x_2, ..., x_0 = x_{k+1}.
\]

Solving the algebraic system (8), we get
\[ x_1 = \frac{bx_{-k}}{a + \sum_{j=0}^{k} \beta_j \prod_{i=0, i \neq j}^{k} x_{-i}} = x_{-k}, \quad (9) \]

then

\[ \frac{bx_{-k}}{a + A} = x_{-k}, \quad (10) \]

i.e.,

\[ |b - (a + A)| x_{-k} = 0 \]

Similarly by the solving the further algebraic equations we get

\[ |b - (a + A)| x_{-k+1} = 0, \]

\[ |b - (a + A)| x_{-k+2} = 0, \]

...,

\[ |b - (a + A)| x_0 = 0. \]

Hence, from assumption we get \( b = (a + A) \). Thus, the proof is completed.

**Example 4.1** In this example we give two different initial data, one of them with period three and the other with period four:

1. The first initial data: \( x_{-1} = 2, x_0 = 5 \) for equation (1) with \( k = 1 \) gives solution of prime period two, Figure 3.

2. The second initial data: \( x_{-5} = 0.1, x_{-4} = 0.6, x_{-3} = 0.2, x_{-2} = 0.5, x_{-1} = 0.3, x_0 = 0.4 \) for equation (1) with \( k = 5 \) gives solution of prime period six, Figure 4.
Figure 3: Prime period two.

Figure 4: Prime period six.
References


