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Fixed Point Theorems In Fuzzy 2-Metric Spaces

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ABSTRACT. In this paper we obtain some fixed point theorems in fuzzy 2- metric spaces on four-mappings and six -mappings using the concept of sub compatible of type A.

1 Introduction

Fixed point theory is an essential part of mathematics because of its application in different areas like variational and linear inequalities, improvement and approximation theory. Many authors proved fixed point theorems in different spaces (see [5-6], [12-17]). Wadkar et al. [3] proved fixed point theorem in dislocated metric space. Zadeh [11] introduced the concept of fuzzy sets which became active field of research for many researchers. Kramosil and Michalek [9], George and Veermani [2] expressed the concept of Fuzzy metric space. Many authors done better work fuzzy metric space and also proved fixed point results in that space. Cho [18] proved the common fixed point theorems in fuzzy metric space, while Chauhan and Utreja [7] proved the common fixed point theorems in fuzzy 2-metric space. The concept of compatibility in fuzzy metric space is introduced by Singh and Chauhan [4] and also proved some common fixed point theorems in that space. Jain and Singh [1], Jungck et al. [8] proved a fixed point theorem for compatible mappings of type (A) in fuzzy metric space. Wadhwa et al. [10]

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introduced the new concepts of sub-compatibility and sub sequential continuity which are respectively weaker than occasionally weak compatibility and reciprocal continuity in Fuzzy metric space using implicit relation and established a common fixed point theorem. Many authors introduced the concept of compatible mappings and proved fixed point theorems.

In this paper we obtain some fixed point theorems in fuzzy 2-metric spaces on four mappings and six-mappings using the concept of sub compatible of type A by sitting an example on six self mappings.

2 Preliminaries

Definition 2.1:A t-norm in $([0,1],*)$ is a binary operation $*:[0,1]\times[0,1]\to[0,1]$ such that the following conditions are satisfied for all $a,b,c,d\in[0,1]$.

1. $a*1=a$,
2. $a*b=b*a$,
3. $a*b\leq c*d$, whenever $a\leq c$ and $b\leq d$,
4. $a*(b*c)=(a*b)*c$.

Definition 2.2: If X is an arbitrary set, * is a continuous t- norm and F is a fuzzy set in $X^2\times[0,\infty)$ which satisfy the following conditions, for all $a,b,c\in X$ and $s,t>0$, then the 3-tuple $(X,F,*)$ is called a fuzzy metric space

1. $F(a,b,0)=0$,
2. $F(a,b,t)=1,\forall t>0$ iff $a=b$,
3. $F(a,b,t)=F(b,a,t)$,
4. $F(a,b,t)*F(b,c,s)\leq F(a,c,t+s)$,
5. $F(a,b,.):[0,\infty)\to[0,1]$ is left continuous,
6. $\lim_{t\to\infty}F(a,b,t)=1$.

Example 2.2a: Let (X,d) be a metric space. We now define $a*b=ab$ or $a*b=mina,b$ and for all $x,y\in X$, $F(x,y,t)=\frac{t}{(t+d(x,y))}$, then the metric d and $(X,F,*)$ respectively are the standard fuzzy metric and a fuzzy metric space.

Definition 2.3: Let $*:[0,1]\times[0,1]\times[0,1]\to[0,1]$ be a binary operation and is said to be a continuous t- norm if following conditions are satisfied:

1. $([0,1],*)$ is an abelian topological monoid with unit 1 and
2. $a_1*b_1*c_1\leq a_2*b_2*c_2$ whenever $a_1\leq a_2,b_1\leq b_2,c_1\leq c_2$ for all a_1,b_1,c_1,a_2,b_2,c_2 in [0,1].

Definition 2.4: If X is an arbitrary set, * is a continuous t- norm and F is a fuzzy set in $X^3\times[0,\infty)$. Then the 3-tuple $(X,F,*)$ is said to be a fuzzy 2- metric space if $\forall a,b,c,u\in X$ and $t_1,t_2,t_3>0$, the following conditions holds.

1. $F(a,b,c,0)=0$,

2. $F(a,b,c,t) = 1, t > 0$ and when out of three points at least two points are equal,
3. $F(a,b,c,t) = F(a,c,b,t) = F(b,c,a,t)$,
4. $F(a,b,c,t_1+t_2+t_3) \geq F(a,b,u,t_1) * F(a,u,c,t_2) * F(u,b,c,t_3)$,
5. $F(a,b,c,.) : [0,\infty) \to [0,1]$ is left continuous.

Definition 2.5: Let $(X,F,*)$ be fuzzy 2-metric space then the sequence $\{x_n\}$ of Fuzzy 2-metric space is convergent to an element $x \in X$ if and only if $\lim\limits_{t\to\infty} F(x_n,x,a,t) = 1$, for all $a \in X$ and $t > 0$.

Definition 2.6: Let $(X,F,*)$ be a fuzzy 2-metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if , $\lim\limits_{t\to\infty} F(x_{n+p},x_n,a,t) = 1$, for all $a \in X, p > 0$ and $t > 0$.

Definition 2.7: If every Cauchy sequence in X is convergent in X then a fuzzy 2- metric space $(X,F,*)$ is complete.

Definition 2.8: Let P and Q be self mappings of a fuzzy 2- metric space $(X,F,*)$. These two mappings are weakly commuting, $F(PQx,QPx,z,t) \geq F(Px,Qx,z,t)\ \forall x \in X$ and $t > 0$.

Definition 2.9: Let $(X,F,*)$ be a fuzzy 2- metric space and P and Q are two self mapping of a fuzzy 2- metric space $(X,F,*)$ then P and Q are compatible if and only if $F(PQx_n,QPx_n,z,t) \to 1$, $\forall t > 0$ whenever the is sequence $\{x_n\}$ of X such that both PQx_n, QPx_n tends to q, for some $q \in X$ as n tends to ∞.

Definition 2.10: Let P and Q be self mappings of a fuzzy 2- metric space $(X,F,*)$. A point $x \in X$ is called a coincidence point of P and Q if and only if $Px = Qx$, then $w = Px = Qx$ is known as a point of coincidence of P and Q.

Definition 2.11: Two self maps P and Q of a fuzzy 2- metric space $(X,F,*)$, if commute at their coincidence points then they are called weakly compatible, i.e. if $Pq = Qq$, for some $q \in X$ then $PQq = QPq$.

Definition 2.12: Consider two self maps P and Q of a fuzzy 2- metric space $(X,F,*)$. These two maps are called as occasionally weakly compatible iff there is a point $x \in X$ which is also coincidence point of P and Q at which they commute.

Definition 2.13: Let $(X,F,*)$ fuzzy 2- metric space with self-mappings P and Q. If there is exist a sequence $\{x_n\}$ in X such that $\lim\limits_{t\to\infty} Px_n = \lim\limits_{t\to\infty} Qx_n = z,\ z \in X$ and satisfy $\lim\limits_{t\to\infty} F(PQx_n,QPx_n,a,t) = 1$ then these two self mappings are called as sub compatible mappings.

Definition 2.14: Self-mappings P and Q of a Fuzzy 2-metric space are said to be sub-compatible of type A if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{t\to\infty} Px_n = \lim\limits_{t\to\infty} Qx_n = z$ and satisfy $\lim\limits_{t\to\infty} F(PQx_n,QQx_n,a,t) = 1$ and $\lim\limits_{t\to\infty} F(PQx_n,PPx_n,a,t) = 1$.

Lemma 2.1: $F(a,b,c,.)$ is non decreasing for all for all $a,b,c \in X$.

Lemma 2.2: Let $(X,M,*)$ be a fuzzy 2-metric space. If there exist $k \in (0,1)$ such that $M(x,y,z,kt) \geq M(x,y,z,t)$, for all $x,y,z \in X$ with $z \neq x, z \neq y$ and $t > 0$ then $x = y$.

3 Main Results

Theorem 3.1: Consider four self mapping $A,B,U\&V$ of a fuzzy 2- metric space $(X,F,*)$ with continuous t-norm defined by $t*t > t$ for all $t \in [0,1]$. If the pairs $(A,U)\&(B,V)$ are sub compatible of type A having the same coincidence points and $AB = BA, BU = UB, UV = VU, AU = UA, AV = VA$, for all $x, y, z \in X, k \in (0, 1)$ and $t > 0$ with

$$F(Ux, Vy, z, kt) \geq Min\{F(Ax, By, z, t), F(Ax, Ux, z, t), F(Vx, By, z, t), F(Ux, Ay, z, t)\}. \quad (1)$$

Then $A, B, U\&V$ have a unique common fixed point in X.

Proof: Since the pairs $(A, U)\&(B, V)$ are sub compatible of type A then there exist two sequences $\{x_n\}\&\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ux_n = a, \ a \in X$ and satisfy
$\lim_{n\to\infty} F(AUx_n, UUx_n, z, t) = 1$, $\lim_{n\to\infty} F(UAx_n, UUx_n, z, t) = 1$.
Thus we have, $\lim_{n\to\infty} F(Aa, Ua, z, t) = 1$, $\lim_{n\to\infty} F(Ua, Aa, z, t) = 1$ and
$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Vy_n = b, \ b \in X$ and satisfy, $\lim_{n\to\infty} F(BVy_n, VVy_n, z, t) = 1$, $\lim_{n\to\infty} F(VBy_n, VVy_n, z, t) = 1$.
Therefore, $Aa = Ua\&Bb = Vb$, where a and b are coincidence points of A, U and B, V respectively. Now we are to prove $a = b$, for this, take $x = x_n$ and $y = y_n$ in (1). Then we get,

$$F(Ux_n, Vy_n, z, kt) \geq Min\{F(Ax_n, By_n, z, t), F(Ax_n, Ux_n, z, t), F(Vy_n, By_n, z, t), F(Ux_n, Ay_n, z, t)\}. \quad (2)$$

By taking limit as $n \to \infty$, we get,

$$F(a, b, z, kt) \geq Min\{F(a, b, z, t), F(a, a, z, t), F(b, b, z, t), F(a, b, z, t)\}.$$

This implies $F(a, b, z, kt) \geq F(a, b, z, t)$ for all $t > 0$.

Hence using lemma 2.2, $a = b$. Which shows that $A, B, U\&V$ have the same coincidence point. Next we are to prove $Aa = Ba = Ua = Va = a$.

First we take $x = a$ and $y = y_n$ in (1), we get,

$$F(Ua, Vy_n, z, kt) \geq Min\{F(Aa, By_n, z, t), F(Aa, Ua, z, t), F(Vy_n, By_n, z, t), F(Ua, Ay_n, z, t)\}.$$

By taking limit as $n \to \infty$, we get,

$$\begin{aligned} F(Ua, b, z, kt) &\geq Min\{F(Aa, b, z, t), F(Aa, Ua, z, t), F(b, b, z, t), F(Ua, b, z, t)\} \\ &\geq Min\{F(Ua, b, z, t), F(Ua, Ua, z, t), F(b, b, z, t), F(Ua, b, z, t)\} \end{aligned} \quad (3)$$

As $a = b$ then $F(Ua, b, z, kt) \geq F(Ua, b, z, t)$. This gives, $Ua = a$.

i.e., $Ua = Aa = a$.

Now we take $x = x_n$ and $y = a$ in (1), we obtain, $F(Ux_n, Va, z, kt) \geq Min\{F(Ax_n, Ba, z, t), F(Ax_n, Ux_n, z, t), F(Va, Ba, z, t), F(Ux_n, Aa, z$

By taking limit as $n \to \infty$, we get,

$$\begin{aligned} F(a, Va, z, kt) &\geq Min\{F(a, Ba, z, t), F(a, a, z, t), F(Va, Ba, z, t), F(a, Aa, z, t)\} \\ &\geq Min\{F(a, a, z, t), F(a, a, z, t), F(Va, a, z, t), F(a, a, z, t)\} \end{aligned} \quad (4)$$

which implies,$F(Va, a, z, kt) \geq F(Va, a, z, t)$.

Which gives, $Va = a$, that is $Va = Ba = a$.

Hence $Aa = Ua = Va = Ba = a$. This completes the proof of the theorem.

Theorem 3.2: Consider six self mappings $A, B, P, Q, S\&T$ of a fuzzy 2- metric space $(X, F, *)$ with continuous t-norm defined by $t * t > t$, for all $t \in [0, 1]$. If the pairs $(AB, S)\&(PQ, T)$ are sub compatible of type A having the same coincidence points and $AB = BA, BS = SB, AS = SA, PQ = QP, TQ = QT, AT = TA, PT = TP$ for all $x, y, z \in X, k \in (0, 1), t > 0$,

$$F(Sx, Ty, z, kt) \geq Min\{F(ABx, PQy, z, t), F(ABx, Sx, z, t), F(Tx, PQy, z, t), F(Sx, ABy, z, t)\}. \tag{5}$$

Then $A, B, P, Q, S\&T$ have a unique common fixed point in X.

Proof: Since the pairs $(AB, S)\&(PQ, T)$ are sub-compatible of type A then there exist two sequences $\{x_n\}\&\{y_n\}$ in X such that $\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} Sx_n = a,\ \ a \in X$ and satisfy
$\lim_{n\to\infty} F(ABSx_n, SSx_n, z, t) = 1$, $\lim_{n\to\infty} F(SABx_n, ABABx_n, z, t) = 1$.
Thus we have, $\lim_{n\to\infty} F(ABa, Sa, z, t) = 1$, $\lim_{n\to\infty} F(Sa, ABa, z, t) = 1$ and
$\lim_{n\to\infty} PQy_n = \lim_{n\to\infty} Ty_n = b,\ \ b \in X$ and satisfy, $\lim_{n\to\infty} F(PQTy_n, TTy_n, z, t) = 1$, $\lim_{n\to\infty} F(TPQy_n, PQPQy_n, z, t) = 1$.
Thus we have, $\lim_{n\to\infty} F(PQb, Tb, z, t) = 1$, $\lim_{n\to\infty} F(Tb, PQb, z, t) = 1$.
Therefore $ABa = Sa\&PQb = Tb$. Thus we have a is coincidence point of $AB\&S$, b is coincidence point of $PQ\&T$.
Now we are to prove $a = b$, for this take $x = x_n\&y = y_n$ in (5), we get

$$F(Sx_n, Ty_n, z, kt) \geq Min\{F(ABx_n, PQy_n, z, t), F(ABx_n, Sx_n, z, t), F(Ty_n, PQy_n, z, t), F(Sx_n, ABy_n, z, t)\}. \tag{6}$$

By taking limit as $n \to \infty$, we get,

$$F(a, b, z, kt) \geq Min\{F(a, b, z, t), F(a, a, z, t), F(b, b, z, t), F(a, b, z, t)\}.$$

This implies $F(a, b, z, kt) \geq F(a, b, z, t)$ for all $t > 0$.
Then by lemma 2.2, $a = b$. This shows that $AB, S, PQ\&T$ have the same coincidence point. Next we are to prove $Aa = Ba = Pa = Qa = Sa = Ta = a$.
First we take $x = a$ and $y = y_n$ in (5), we get,

$$F(Sa, Ty_n, z, kt) \geq Min\{F(ABa, PQy_n, z, t), F(ABa, Sa, z, t), F(Ty_n, PQy_n, z, t), F(Sa, ABy_n, z, t)\}.$$

By taking limit as $n \to \infty$, we get,

$$F(Sa, b, z, kt) \geq Min\{F(ABa, b, z, t), F(ABa, Sa, z, t), F(b, b, z, t), F(Sa, b, z, t)\} \tag{7}$$

As $a = b$ then $F(Sa, a, z, kt) \geq F(Sa, a, z, t)$. This gives, $Sa = a$.
Now we take $x = x_n$ and $y = a$ in (5), we obtain, $F(Sx_n, Ta, z, kt) \geq Min\{F(ABx_n, PQa, z, t), F(ABx_n, Sx_n, z, t), F(Ta, PQa, z, t), F(Sx_n,$
By taking limit as $n \to \infty$, we get,

$$\begin{aligned} F(a, Ta, z, kt) &\geq Min\{F(a, PQa, z, t), F(a, a, z, t), F(Ta, PQa, z, t), F(a, ABa, z, t)\} \\ &\geq Min\{F(a, Ta, z, t), F(a, a, z, t), F(Ta, Ta, z, t), F(a, Sa, z, t)\} \\ &\geq F(Ta, a, z, t) \end{aligned} \tag{8}$$

Which gives, $Ta = a$. Now we prove $Aa = Ba = a$.
Put $x = Ba$ and $y = y_n$in (5), we get,
$F(SBa, Ty_n, z, kt) \geq Min\{F(ABBa, PQy_n, z, kt), F(ABBa, SBa, z, t), F(Ty_n, PQy_n, z, t), F(SBa, ABy_n, z, t)\}$. As A, B

and S commutes $ABBa = BABa = BSa = Ba$.

$$\begin{aligned} F(Ba, a, z, kt) &\geq Min\{F(Ba, a, z, t), F(Ba, Ba, z, t), F(a, a, z, t), F(Ba, a, z, t)\} \\ &\geq F(Ba, a, z, t) \end{aligned} \tag{9}$$

Therefore $Ba = a$.

Now put $x = Aa$ and $y = y_n$in (5), we get,

$F(SAa, Ty_n, z, kt) \geq Min\{F(ABAa, PQy_n, z, t), F(ABAa, SAa, z, t), F(Ty_n, PQy_n, z, t), F(SAa, ABy_n, z, t)\}$. As A, B and S commutes $ABAa = ASa = Aa$ $SAa = ASa = Aa$.

$F(Aa, PQy_n, z, kt) \geq Min\{F(Aa, PQy_n, z, t), F(Aa, Aa, z, t), F(Ty_n, PQy_n, z, t), F(Aa, ABy_n, z, t)\}$

$$\begin{aligned} F(Aa, a, z, kt) &\geq Min\{F(Aa, a, z, t), F(Aa, Aa, z, t), F(a, a, z, t), F(Aa, a, z, t)\} \\ &\geq F(Aa, a, z, t) \end{aligned} \tag{10}$$

Thus $Aa = a$. Hence we have $Aa = Ba = Sa = a$. Similarly to prove $Qa = a$, take $x = x_n$ and $y = Qa$ and to prove $Pa = a$, take $x = x_n$ and $y = Pa$. Hence we have $Aa = Ba = Pa = Qa = Sa = Ta = a$.

Example 2.3: Let A, B, S, T, P and Q be self mappings of $X = [0, 1]$, where $Bx = \frac{x}{2}, Ax = \frac{x}{3}, Sx = \frac{x}{6}, Tx = \frac{x}{6}, Qx = 2x$ and $Px = \frac{x}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences, where $x_n = \frac{n}{n+1}, y_n = \frac{n^2}{n^2+1}$. Then $\frac{1}{6}$ is a fixed point of A, B, S, T, P and Q.

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