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Semiinfinite multiobjective fractional programming problems using exponential type generalized invexities

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ABSTRACT. In this paper, first a class of second order exponential type hybrid $(\alpha, \beta, \gamma, \eta, \rho, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \theta)$-invexities is introduced, and then a class of parametrically sufficient efficiency conditions based on the second order exponential type hybrid invexities is established. Finally some parametric sufficient efficiency theorems under the higher order exponential type hybrid $(\alpha, \beta, \gamma, \eta, \rho, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \theta)$-invexities are investigated to the context of solving a semiinfinite multiobjective fractional programming problem. The notions of the second order exponential type hybrid $(\alpha, \beta, \gamma, \eta, \rho, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \theta)$-invexities are new and encompass most of the generalized invexity concepts in the literature. To the best of our knowledge, the results on semiinfinite multiobjective fractional programming problems established in this paper are new and application-oriented toward multitime multiobjective problems as well as multiobjective control problems.

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1 Introduction

In recent publications, Zalmai [52] introduced the Hanson-Antczak-type generalized $(\alpha, \beta, \gamma, \eta, \rho, \theta)$-V-invexities based on the class of V-r-invex functions defined by Antczak [1], and applying these new functions, established a number of parametric sufficient efficiency results under various Hanson-Antczak-type generalized $(\alpha, \beta, \gamma, \rho, \eta, \theta)$-V-invexity frameworks for the semiinfinite multiobjective fractional programming problems. Verma [41 - 44] has investigated some results on the multiobjective fractional programming based on new $\epsilon-$efficiency conditions, and second-order $(\Phi, \eta, \rho, \theta)$-invexities for parameter-free $\epsilon-$efficiency conditions. On the other hand, Pitea

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and Postolache [35 - 37], motivated by the significant applications to mechanical engineering (where curvilinear integral type objectives are extensively applied due to their physical meaning as mechanical work), introduced a new class of multitime multiobjective variational problems of minimizing a vector of functionals of curvilinear integral type, and applied to establish certain new conditions for Mond-Weir-Zalmai type duality for multitime multiobjective variational problems using the notion of the (ρ, b)-quasiinvexity. The curvilinear integral type objectives play an essential role in mathematical modeling of certain processes relating to robotics, tribology, and others. This is equivalent to stating that for a given number of r sources producing mechanical work, minimize r on a set of limited resources. There are also accelerated advances investigating duality for a class of multiobjective control problems based on the generalized invexity by Zhian and Qingkai [55], Mond and Smart [32], Bhatia and Kumar [4], where the objective functionals and constraint functionals are different as well as the same. Zhian and Qingkai [55] have applied duality models to more generalized aspects for the Mond-Smart generalized invexity [32] to the context of scalar control problems. Verma [42] established a class of results for multiobjective fractional subset programming problems as well.

Next, we introduce the differentiable functions $h, \kappa : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $\omega, \varpi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which are crucial to the higher order exponential type hybrid (α, β, γ, η, ρ, $h(\cdot,\cdot)$, $\kappa(\cdot,\cdot)$, $\omega(\cdot,\cdot,\cdot)$, $\varpi(\cdot,\cdot,\cdot)$, θ)-invexity framework. Based on this new framework for the higher order exponential type hybrid (α, β, γ, η, ρ, $h(\cdot,\cdot)$, $\kappa(\cdot,\cdot)$, $\omega(\cdot,\cdot,\cdot)$, $\varpi(\cdot,\cdot,\cdot)$, θ)-invexities, we consider the following semiinfinite multiobjective fractional programming problem:

$$(P) \qquad \text{Minimize } \varphi(x) = \big(\varphi_1(x), \dots, \varphi_p(x)\big) = \left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right)$$

subject to

$$G_j(x,t) \leqq 0 \text{ for all } t \in T_j, \quad j \in \underline{q},$$

$$H_k(x,s) = 0 \text{ for all } s \in S_k, \quad k \in \underline{r},$$

$$x \in X,$$

where p, q, and r are positive integers, X is a nonempty open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space) for each $j \in \underline{q} \equiv \{1, 2, \dots, q\}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces for each $i \in \underline{p}$, f_i and g_i are real-valued functions defined on X for each $j \in \underline{q}$, $G_j(\cdot, t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $H_k(\cdot, s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $G_j(x, \cdot)$ and $H_k(x, \cdot)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

We observe that multiobjective programming problems of the form (P) but with a finite number of constraints (where the functions G_j are independent of t, and the functions H_k are independent of s) have been investigated for the past several decades with several classes of static and dynamic optimization problems with multiple fractional objective functions that have been considered leading to a number of sufficient efficiency and duality results currently available in the related literature. In this communication, we intend first to introduce the higher order exponential type hybrid $(\alpha, \beta, \gamma, \rho, \eta, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \theta)$-invexities, and then we formulate a number of parametric sufficient efficiency results for the semiinfinite multiobjective fractional programming problem (P) under various generalized higher order exponential type hybrid $(\alpha, \beta, \gamma, \rho, \eta, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \theta)$-invexity frameworks. The semiinfinite programming problems (if it has a finite number of variables and infinitely many constraints) offer more constructive applications in terms of theoretical as well as concrete, real-world problems, including probability and statistics, engineering design, boundary value problems, defect minimization for operator equations, geometry, random graphs, wavelet analysis, reliability testing, environmental protection planning, semidefinite programming, optimal control problems, and robotics. Problems of this nature have been applied for the modeling and analysis of a wide range of theoretical as well as concrete, real-world, practical problems. Semiinfinite programming concepts and techniques have found relevance and applications in approximation theory, statistical models, game theory, engineering design, boundary value problems, graphs related to Newton flows, wavelet analysis, reliability testing, environmental protection planning, decision making and management, geometric programming, optimal control problems, robotics, and continuum mechanics, among others. For more details, we refer the reader [1 - 55].

We observe that all the parametrically sufficient efficiency results established in this paper can easily be modified and restated for each one of the following classes of nonlinear programming problems, which are special cases of (P):

$(P1)$ $\quad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \big(f_1(x),\dots,f_p(x)\big),$

$(P2)$ $\quad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \dfrac{f_1(x)}{g_1(x)},$

$(P3)$ $\quad \underset{x\in\mathbb{F}}{\text{Minimize}} \ f_1(x),$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x,t) \leqq 0 \ \text{ for all } t \in T_j,\ j \in \underline{q}, H_k(x,s) = 0 \ \text{ for all } s \in S_k,\ \ k \in \underline{r}\},$$

$(P4)$ $\quad$ Minimize $\left(\dfrac{f_1(x)}{g_1(x)},\dots,\dfrac{f_p(x)}{g_p(x)}\right)$

subject to

$$\tilde{G}_j(x) \leqq 0,\ \ j \in \underline{q}, \quad \tilde{H}_k(x) = 0,\ \ k \in \underline{r},\ \ x \in X,$$

where f_i and g_i, $i \in \underline{p}$, are as defined in the description of (P), $\tilde{G}_j$, $j \in \underline{q}$, and $\tilde{H}_k$, $k \in \underline{r}$, are real-valued functions defined on X,

$(P5)$ $\quad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \big(f_1(x),\dots,f_p(x)\big),$

$(P6)$ $\quad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \dfrac{f_1(x)}{g_1(x)},$

$(P7)$ $\quad \underset{x\in\mathbb{G}}{\text{Minimize}} \ f_1(x),$

where $\mathbb{G}$ is the feasible set of $(P4)$, that is,

$$\mathbb{G} = \{x \in X : \tilde{G}_j(x) \leqq 0,\ j \in \underline{q},\quad \tilde{H}_k(x) = 0,\ \ k \in \underline{r}\}.$$

The paper begins with an introductory section, while in Section 2, the higher order exponential type hybrid (α, β, γ, η, ρ, $h(\cdot,\cdot)$, $\kappa(\cdot,\cdot)$, $\omega(\cdot,\cdot,\cdot)$, $\varpi(\cdot,\cdot,\cdot)$, θ)-invexities, which generalize $HA(\alpha,\beta,\gamma,\rho,\eta,\theta)$-V-invexities introduced and studied by Zalmai [52], are introduced, while encompass most of the invexity notions in the literature. In Section 3, we present some sufficient efficiency conditions, and prove several sets of sufficiency criteria under a variety of the higher order exponential type hybrid $(\alpha,\beta,\gamma,\rho,\eta,h(\cdot,\cdot),\theta)$-invexities that are placed on certain vector-valued functions whose entries consist of the individual as well as some combinations of the problem functions. We also observe that all the parametric sufficient efficiency results established in this paper regarding problem (P) can be modified to several special classes of nonlinear programming problems.

2 Preliminaries

In this section we first introduce the notion of the higher order exponential type hybrid (α, β, γ, η, ρ, $h(\cdot,\cdot)$, $\kappa(\cdot,\cdot)$, $\omega(\cdot,\cdot,\cdot)$, $\varpi(\cdot,\cdot,\cdot)$, θ)-invexities, and then recall some other related auxiliary results instrumental to the problem on hand. Recently, Antczak [1] introduced the following variant of the class of V-invex functions.

Definition 2.1 *Let X be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : X \to \mathbb{R}^k$ is called (strictly) $\zeta_i - \tilde{r}$-invex at $u \in X$ if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\zeta_i : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $i \in \underline{k}$ such for each $x \in X$,*

$$\frac{1}{\tilde{r}} e^{\tilde{r} f_i(x)} (>) \geqq \frac{1}{\tilde{r}} e^{\tilde{r} f_i(u)} [1 + \tilde{r}\zeta_i(x,u)\langle \nabla f_i(u), \eta(x,u)\rangle] \ \ \textit{for } \tilde{r} \neq 0,$$

$$f_i(x) - f_i(u) \geqq \zeta_i(x,u)\langle \nabla f_i(u), \eta(x,u)\rangle \ \ \textit{for } \tilde{r} = 0.$$

This class of functions was considered in [1] for establishing some sufficiency and duality results for a nonlinear programming problem with differentiable functions, and their nonsmooth analogs were discussed in [2]. Recently, Zalmai [52] introduced the Hanson-Antczak type generalized $HA(\alpha,\beta,\gamma,\eta,\rho,\theta)$-V-invexity, an exponential type framework, and then he applied to a set of problems on fractional programming. As a result, he further envisioned a vast array of interesting and significant classes of generalized convex functions. Now inspired by [52], we present higher order exponential type hybrid (α, β, γ, η, $h(\cdot,\cdot)$, ρ, θ)-invexities that generalize and encompass most of the existing notions available in the current literature. Let the function $f = (f_1, f_2, \ldots, f_p) : X \to \mathbb{R}^p$ be differentiable at x^*.

Definition 2.2 *Let X be a nonempty open convex subset of $\mathbb{R}^n$. The function f is said to be (strictly) higher order exponential type hybrid (α, β, γ, $h(\cdot,\cdot)$, η, ρ, θ)-invex at $x^* \in X$ if there exist functions $\alpha : X \times X \to \mathbb{R}$, $\beta : X \times X \to \mathbb{R}$, $\gamma_i : X \times X \to \mathbb{R}_+$, $h_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $i \in \underline{p}$, $z \in \mathbb{R}^n$, $\rho_i : X \times X \to \mathbb{R}$, $i \in \underline{p}$, $\eta : X \times X \to \mathbb{R}^n$ and $\theta : X \times X \to \mathbb{R}^n$*

such that for all $x \in X$ $(x \neq x^*)$ *and* $i \in \underline{p}$,

$$\frac{1}{\alpha(x,x^*)}\gamma_i(x,x^*)\Big(e^{\alpha(x,x^*)[f_i(x)-f_i(x^*)]}-1\Big)(>) \geqq \frac{1}{\beta(x,x^*)}\Big([\langle \nabla_z h_i(x^*,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\rangle] + [\langle h_i(x^*,z), e^{\beta(x,x^*)}-1\rangle] - [\langle \nabla_z h_i(x^*,z), e^{\beta(x,x^*)z}-\mathbf{1}\rangle] + \rho_i(x,x^*)\|\theta(x,x^*)\|^2 \quad \text{if } \alpha(x,x^*) \neq 0 \text{ and } \beta(x,x^*) \neq 0 \text{ for all } x \in X,$$

where $\|\cdot\|$ *is a norm on* $\mathbb{R}^n$ *and*

$$\big(e^{\beta(x,x^*)\eta(x,x*)}-\mathbf{1}\big) \equiv \big(e^{\beta(x,x^*)\eta_1(x,x^*)}-1,\ldots,e^{\beta(x,x^*)\eta_n(x,x^*)}-1\big).$$

Definition 2.3 *Let X be a nonempty open convex subset of* $\mathbb{R}^n$. *The function f is said to be (strictly) higher order exponential type hybrid* (α, β, γ, η, ρ, $h(\cdot,\cdot)$, θ)*-pseudoinvex at* $x^* \in X$ *if there exist functions* $\alpha: X\times X \to \mathbb{R}$, $\beta: X\times X\to\mathbb{R}$, $\gamma: X\times X\to\mathbb{R}_+$, $i\in\underline{p}$, $z\in\mathbb{R}^n$, $\rho: X\times X\to\mathbb{R}$, $\eta: X\times X\to\mathbb{R}^n$ *and* $\theta: X\times X\to\mathbb{R}^n$, $h_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ $i\in\underline{p}$, *such that for all* $x\in X$ $(x\neq x^*)$,

$$\frac{1}{\beta(x,x^*)}\Big(\Big\langle \sum_{i=1}^{p}\nabla_z h_i(x^*,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\Big\rangle + [\langle h_i(x^*,z), e^{\beta(x,x^*)}-\mathbf{1}\rangle] - [\langle \nabla_z h_i(x^*,z), e^{\beta(x,x^*)z}-\mathbf{1}\rangle]\Big) \geqq -\rho(x,x^*)\|\theta(x,x^*)\|^2$$
$$\Rightarrow \quad \frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\Big(e^{\alpha(x,x^*)\sum_{i=1}^{p}[f_i(x)-f_i(x^*)]}-1\Big)(>)\geqq 0$$
$$\text{if } \alpha(x,x^*)\neq 0 \text{ and } \beta(x,x^*)\neq 0 \text{ for all } x\in X.$$

Definition 2.4 *Let X be a nonempty open convex subset of* $\mathbb{R}^n$. *The function f is said to be (prestrictly) higher order exponential type hybrid* (α, β, γ, η, ρ, $h(\cdot,\cdot)$, θ)*-quasiinvex at* $x^*\in X$ *if there exist functions* $\alpha: X\times X\to\mathbb{R}$, $\beta: X\times X\to\mathbb{R}$, $\gamma: X\times X\to\mathbb{R}_+$, $i\in\underline{p}$, $\rho: X\times X\to\mathbb{R}$, $\eta: X\times X\to\mathbb{R}^n$ *and* $\theta: X\times X\to\mathbb{R}^n$, $h_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ $i\in\underline{p}$, *such that for all* $x\in X$, *and for* $z\in\mathbb{R}^n$,

$$\frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\Big(e^{\alpha(x,x^*)\sum_{i=1}^{p}[f_i(x)-f_i(x^*)]}-1\Big)(<)\leqq 0$$
$$\Rightarrow \quad \frac{1}{\beta(x,x^*)}\Big(\Big\langle \sum_{i=1}^{p}\nabla_z h_i(x^*,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\Big\rangle + [\langle h_i(x^*,z), e^{\beta(x,x^*)}-1\rangle] - [\langle \nabla_z h_i(x^*,z), e^{\beta(x,x^*)z}-\mathbf{1}\rangle]\Big) \leqq -\rho(x,x^*)\|\theta(x,x^*)\|^2 \text{if } \alpha(x,x^*)\neq 0 \text{ and } \beta(x,x^*)\neq 0 \text{ for all } x\in X.$$

Next, we present a special case in which our generalized higher order exponential type (α, β, γ, η, ρ, $h(\cdot,\cdot)$, θ)-invexity reduces to a similar form (Zalmai [52]) if we set

$$h(x^*,z) = \langle z, \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z\rangle,$$

where the function $f = (f_1, f_2, \dots, f_p) : X \to \mathbb{R}^p$ is differentiable at x^*.

Example 2.1 *Let X be a nonempty open convex subset of $\mathbb{R}^n$. Consider functions $\alpha : X \times X \to \mathbb{R}$, $\beta : X \times X \to \mathbb{R}$, $\gamma : X \times X \to \mathbb{R}_+$, $\xi : X \times X \to \mathbb{R}_+\backslash\{0\}$, $i \in \underline{p}$, $\rho : X \times X \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, and $\theta : X \times X \to \mathbb{R}^n$ for all $x \in X\,(x \neq x^*)$, where $x^* \in X$ and $z \in \mathbb{R}^n$. Then the function f is (strictly) higher order exponential type hybrid (α, β, η, γ, ξ, ρ, θ)-invex at $x^* \in X$ if*

$$\frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\Big(e^{\alpha(x,x^*)[f_i(x)-f_i(x^*)]}-1\Big)(>)$$
$$\geqq \frac{1}{\beta(x,x^*)}\Big[\xi(x,x^*)\big[\big\langle \nabla f(x^*)+\nabla^2 f(x^*)z, e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\big\rangle$$
$$-\frac{1}{2}\big\langle \nabla^2 f(x^*)z, e^{\beta(x,x^*)z}-\mathbf{1}\big\rangle\big]\Big]+\rho_i(x,x^*)\|\theta(x,x^*)\|^2$$
$$\text{if } \alpha(x,x^*)\neq 0 \text{ and } \beta(x,x^*)\neq 0 \text{ for all } x\in X,$$

and

$$\gamma(x,x^*)[f_i(x)-f_i(x^*)]$$
$$(>)\geqq \xi(x,x^*)\big[\big\langle \nabla f(x^*)+\nabla^2 f(x^*)z, \eta(x,x^*)\big\rangle$$
$$-\frac{1}{2}\big\langle \nabla^2 f(x^*)z, z\big\rangle\big]+\rho_i(x,x^*)\|\theta(x,x^*)\|^2$$
$$\text{if } \alpha(x,x^*)= 0 \text{ and } \beta(x,x^*)= 0 \text{ for all } x\in X.$$

We also observe that for the proofs of the sufficient efficiency theorems, sometimes it may be more convenient to apply certain alternative but equivalent forms of the above definitions based on considering the contrapositive statements. For example, the higher order exponential type hybrid (α, β, γ, η, ρ, $h(\cdot,\cdot)$, θ)-V-quasiinvexity (when $\alpha(x,x^*)\neq 0$ and $\beta(x,x^*)\neq 0$ for all $x\in X$) can be defined in the following equivalent way:

The function f is a higher order exponential type hybrid (α, β, γ, η, ρ, $h(\cdot,\cdot)$, θ)-quasiinvex at $x^*\in X$ if there exist functions $\alpha : X\times X\to\mathbb{R}$, $\beta : X\times X\to\mathbb{R}$, $\gamma : X\times X\to\mathbb{R}_+$, $i\in\underline{p}$, $\rho : X\times X\to\mathbb{R}$, $\theta : X\times X\to\mathbb{R}^n$, and $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ (differentiable) such that for all $x\in X$ and $z\in\mathbb{R}^n$,

$$\frac{1}{\beta(x,x^*)}\Big(\Big\langle \sum_{i=1}^{p}\nabla_z h_i(x^*,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\Big\rangle+[\langle h_i(x^*,z), e^{\beta(x,x^*)}-1\rangle]$$
$$-[\langle \nabla_z h_i(x^*,z), e^{\beta(x,x^*)z}-\mathbf{1}\rangle]\Big) > -\rho(x,x^*)\|\theta(x,x^*)\|^2$$
$$\Rightarrow \frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\Big(e^{\alpha(x,x^*)\sum_{i=1}^{p}\xi_i(x,x^*)[f_i(x)-f_i(x^*)]}-1\Big)>0.$$

In the sequel, we introduce a consistent notation for vector inequalities. For $a,b\in\mathbb{R}^m$, the following order notation will be used: $a\geqq b$ if and only if $a_i\geqq b_i$ for all $i\in\underline{m}$; $a\geqslant b$ if and only if $a_i\geqq b_i$ for all $i\in\underline{m}$, but $a\neq b$;

$a > b$ if and only if $a_i > b_i$ for all $i \in \underline{m}$; and $a \ngeqslant b$ is the negation of $a \geqslant b$.

For the purpose of comparison with the sufficient efficiency conditions that will be proposed and discussed in this paper, we next recall a set of necessary efficiency conditions for (P).

Theorem 2.1 [52] *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $G_j(\cdot, t)$ be continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $H_k(\cdot, s)$ be continuously differentiable at x^* for all $s \in S_k$. If x^* is an efficient solution of (P), if the generalized Guignard constraint qualification holds at x^*, and if for each $i_0 \in \underline{p}$, the set cone$\big(\{\nabla G_j(x^*, t) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \cup \{\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*) : i \in \underline{p}, i \neq i_0\}\big) + span\big(\{\nabla H_k(x^*, s) : s \in S_k, k \in \underline{r}\}\big)$ is closed, then there exist $u^* \in U$ and integers ν_0^* and ν^*, with $0 \leqq \nu_0^* \leqq \nu^* \leqq n+1$, such that there exist ν_0^* indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0^*}$, $\nu^* - \nu_0^*$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*} \backslash \underline{\nu_0^*}$, and ν^* real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that*

$$\sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_{k_m}(x^*, s^m) = 0,$$

where cone(V) is the conic hull of the set $V \subset \mathbb{R}^n$ (i.e., the smallest convex cone containing V), span(V) is the linear hull of V (i.e., the smallest subspace containing V), $\hat{T}_j(x^*) = \{t \in T_j : G_j(x^*, t) = 0\}$, $U = \{u \in \mathbb{R}^p : u > 0, \sum_{i=1}^{p} u_i = 1\}$, and $\underline{\nu^*} \backslash \underline{\nu_0^*}$ is the complement of the set $\underline{\nu_0^*}$ relative to the set $\underline{\nu^*}$.

3 Sufficient Efficiency Conditions

In this section, we present several sets of sufficiency results in which various generalized higher order exponential type hybrid $(\alpha, \beta, \gamma, \eta, \rho, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \theta)$-invexity assumptions are imposed on certain vector functions whose components are the individual as well as some combinations of the problem functions, where $h, \kappa : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $\omega, \varpi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are differentiable functions. Let the function $\mathcal{E}_i(\cdot, \lambda, u) : X \to \mathbb{R}$ be defined, for fixed λ and u, on a nonempty open convex subset X of $\mathbb{R}^n$ by

$$\mathcal{E}_i(z, \lambda, u) = u_i[f_i(z) - \lambda_i g_i(z)],\ i \in \underline{p}.$$

Theorem 3.1 *Let $x^* \in \mathbb{F}$, $\lambda^* = \varphi(x^*)$, the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$ and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$, $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν with $0 \leqq \nu_0 \leqq \nu \leqq n+1$ such that there exist ν_0 indices j_m with $1 \leqq j_m \leqq q$ together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m with $1 \leqq k_m \leqq r$ together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$ for each critical direction $z \in \mathbb{R}^n$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$ and with the property*

$$\sum_{i=1}^{p} u_i^* [\nabla_z h_i(x^*, z) - \lambda_i^* \nabla_z \kappa_i(x^*, z)] + \sum_{m=1}^{\nu_0} v_m^* \nabla_z \omega_{j_m}(x^*, t^m, z) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla_z \varpi_{k_m}(x^*, s^m, z) = 0. \tag{1}$$

$$\sum_{i=1}^{p} u_i^*[h_i(x^*,z) - \lambda_i^* \kappa_i(x^*,z)] + \sum_{m=1}^{\nu_0} v_m^* \omega_{j_m}(x^*,t^m,z) + \sum_{m=\nu_0+1}^{\nu} v_m^* \varpi_{k_m}(x^*,s^m,z) = 0. \quad (2)$$

Assume, further that either one of the following two sets of conditions holds:

(a) *(i)* *f_i is higher order exponential type hybrid $(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, h(\cdot,\cdot), \theta)$-invex at x^*, g_i is higher order exponential type hybrid $(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \kappa(\cdot,\cdot), \theta)$-invex at x^*, and $\bar{\gamma}(x,x^*) > 0$ for all $x \in \mathbb{F}$;*

(ii) *$\left(v_1^* G_{j_1}(\cdot,t^1), \dots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot,t^{\nu_0})\right)$ is higher order exponential type hybrid $(\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \omega(\cdot,\cdot,\cdot), \theta)$-invex at x^*;*

(iii) *$\left(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}), \dots, v_\nu^* H_{k_\nu}(\cdot,s^\nu)\right)$ is higher order exponential type hybrid $(\alpha, \beta, \breve{\gamma}, \eta, \breve{\rho}, \varpi(\cdot,\cdot,\cdot), \theta)$-invex at x^*;*

(iv) *$\sum_{i=1}^{p} u_i^* \bar{\rho}_i(x,x^*) + \sum_{m=1}^{\nu_0} \hat{\rho}_m(x,x^*) + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x,x^*) \geqq 0$ for all $x \in \mathbb{F}$;*

(b) the function $\left(L_1(\cdot,u^*,v^*,\lambda^*,\bar{t},\bar{s}), \dots, L_p(\cdot,u^*,v^*,\lambda^*,\bar{t},\bar{s})\right)$ is higher order exponential type hybrid $(\alpha, \beta, \gamma, 0, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \theta)$-pseudoinvex at x^* and $\gamma(x,x^*) > 0$ for all $x \in \mathbb{F}$, where

$$L_i(z,u^*,v^*,\lambda^*,\bar{t},\bar{s}) = u_i^*\Big[f_i(z) - \lambda_i^* g_i(z) + \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(z,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(z,s^m)\Big], \quad i \in \underline{p}.$$

Then x^* is an efficient solution to (P).

(a) Based on our assumptions in (i) - (iii), we have

$$\frac{1}{\alpha(x,x^*)}\bar{\gamma}_i(x,x^*)\Big(e^{\alpha(x,x^*)\{f_i(x)-\lambda_i^* g_i(x)-[f_i(x^*)-\lambda_i^* g_i(x^*)]\}} - 1\Big) \geqq \frac{1}{\beta(x,x^*)}\Big(\Big\langle [\nabla_z h_i(x^*,z) - \lambda_i^* \nabla_z \kappa_i(x^*,z)], e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1}\Big\rangle + \Big\langle [h_i(x^*,z) - \lambda_i^* \kappa_i(x^*,z)], e^{\beta(x,x^*)} - 1\Big\rangle - \Big\langle [\nabla_z h_i(x^*,z) - \lambda_i^* \nabla_z \kappa_i(x^*,z)], e^{\beta(x,x^*)z} - \mathbf{1}\Big\rangle\Big) + \bar{\rho}_i(x,x^*)\|\theta(x,x^*)\|^2, \ i \in \underline{p}, \quad (3)$$

$$\frac{1}{\alpha(x,x^*)}\hat{\gamma}_m(x,x^*)\Big(e^{\alpha(x,x^*)[v_m^* G_{j_m}(x,t^m) - v_m^* G_{j_m}(x^*,t^m)]} - 1\Big) \geqq \frac{1}{\beta(x,x^*)}\Big(\Big\langle v_m^* \nabla_z \omega_{j_m}(x^*,t^m,z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1}\Big\rangle + \Big\langle v_m^* \omega_{j_m}(x^*,t^m,z), e^{\beta(x,x^*)} - 1\Big\rangle - \Big\langle v_m^* \nabla_z \omega_{j_m}(x^*,t^m,z), e^{\beta(x,x^*)z} - 1\Big\rangle\Big) + \hat{\rho}_m(x,x^*)\|\theta(x,x^*)\|^2, \ m \in \underline{\nu_0}, \quad (4)$$

$$\frac{1}{\alpha(x,x^*)}\breve{\gamma}_m(x,x^*)\Big(e^{\alpha(x,x^*)[v_m^* H_{k_m}(x,s^m)-v_m^* H_{k_m}(x^*,s^m)]}-1\Big)$$
$$\geqq \frac{1}{\beta(x,x^*)}\Big(\Big\langle v_m^*\nabla_z\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\Big\rangle$$
$$+\Big\langle v_m^*\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)}-1\Big\rangle$$
$$-\Big\langle v_m^*\nabla_z\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)z}-\mathbf{1}\Big\rangle\Big)$$
$$+\breve{\rho}_m(x,x^*)\|\theta(x,x^*)\|^2,\ m\in\underline{\nu}\backslash\underline{\nu_0}. \quad (5)$$

Multiplying (3.3) by u_i^* and then summing over $i \in \underline{p}$, summing (3.4) over $m \in \underline{\nu_0}$, and summing (3.5) over $m \in \underline{\nu}\backslash\underline{\nu_0}$, and finally adding the resulting inequalities, we get

$$\frac{1}{\alpha(x,x^*)}\Big\{\sum_{i=1}^{p}u_i^*\bar{\gamma}_i(x,x^*)\Big(e^{\alpha(x,x^*)\{f_i(x)-\lambda_i^* g_i(x)-[f_i(x^*)-\lambda_i^* g_i(x^*)]\}}-1\Big)$$
$$+\sum_{m=1}^{\nu_0}\hat{\gamma}_m(x,x^*)\Big(e^{\alpha(x,x^*)[v_m^* G_{j_m}(x,t^m)-v_m^* G_{j_m}(x^*,t^m)]}-1\Big)$$
$$+\sum_{m=\nu_0+1}^{\nu}\breve{\gamma}_m(x,x^*)\Big(e^{\alpha(x,x^*)[v_m^* H_{k_m}(x,s^m)-v_m^* H_{k_m}(x^*,s^m)]}-1\Big)\Big\}$$
$$\geqq \frac{1}{\beta(x,x^*)}\Big(\Big\langle \sum_{i=1}^{p}u_i^*[\nabla_z h_i(x^*,z)-\lambda_i^*\nabla_z\kappa_i(x^*,z)]+\sum_{m=1}^{\nu_0}v_m^*\nabla_z\omega_{j_m}(x^*,t^m,z)$$
$$+\sum_{m=\nu_0+1}^{\nu}v_m^*\nabla_z\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\Big\rangle$$
$$+\Big\langle \sum_{i=1}^{p}u_i^*[h_i(x^*,z)-\lambda_i^*\kappa_i(x^*,z)]+\sum_{m=1}^{\nu_0}v_m^*\omega_{j_m}(x^*,t^m,z)$$
$$+\sum_{m=\nu_0+1}^{\nu}v_m^*\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)}-1\Big\rangle$$
$$-\Big\langle \sum_{i=1}^{p}u_i^*[\nabla_z h_i(x^*,z)-\lambda_i^*\nabla_z\kappa_i(x^*,z)]+\sum_{m=1}^{\nu_0}v_m^*\nabla_z\omega_{j_m}(x^*,t^m,z)$$
$$+\sum_{m=\nu_0+1}^{\nu}v_m^*\nabla_z\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)z}-\mathbf{1}\Big\rangle\Big)$$
$$+\Big[\sum_{i=1}^{p}u_i^*\bar{\rho}_i(x,x^*)+\sum_{m=1}^{\nu_0}\hat{\rho}_m(x,x^*)+\sum_{m=\nu_0+1}^{\nu}\breve{\rho}_m(x,x^*)\Big]\|\theta(x,x^*)\|^2.$$

Next using (3.1), (3.2), (iv), $\varphi(x^*) = \lambda^*;\ x,\ x^* \in \mathbb{F}$, and $G_{j_m}(x^*,t^m) = 0$ for all $m \in \underline{\nu_0}$, the above inequality reduces to

$$\frac{1}{\alpha(x,x^*)}\sum_{i=1}^{p}u_i^*\bar{\gamma}_i(x,x^*)\Big(e^{\alpha(x,x^*)[f_i(x)-\lambda_i^* g_i(x)]}-1\Big)\geqq 0.$$

Since $\gamma(x,x^*) > 0$, even if we consider the both cases $\alpha(x,x^*) > 0$ and $\alpha(x,x^*) < 0$, it follows from the above inequality

$$\sum_{i=1}^{p}u_i^*[f_i(x)-\lambda_i^* g_i(x)]\geqq 0. \quad (6)$$

Therefore, we conclude that x^* is an efficient solution of (P).

(b): Let x be an arbitrary feasible solution of (P). From (3.1)and (3.2), respectively, we arrive at

$$\frac{1}{\beta(x,x^*)}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla_z h_i(x^*,z) - \lambda_i^* \nabla_z \kappa_i(x^*,z)] + \sum_{m=1}^{\nu_0} v_m^* \nabla_z \omega_{j_m}(x^*,t^m,z) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla_z \varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1}\Big\rangle \geq 0$$

and

$$\frac{1}{\beta(x,x^*)}\Big\langle \sum_{i=1}^{p} u_i^*[h_i(x^*,z) - \lambda_i^* \kappa_i(x^*,z)] + \sum_{m=1}^{\nu_0} v_m^* \omega_{j_m}(x^*,t^m,z) + \sum_{m=\nu_0+1}^{\nu} v_m^* \varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)} - \mathbf{1}\Big\rangle \geq 0.$$

Thus, applying the $(\alpha,\beta,\gamma,0,h(\cdot,\cdot),\kappa(\cdot,\cdot),\omega(\cdot,\cdot,\cdot),\varpi(\cdot,\cdot,\cdot),\theta)$-pseudoinvexity assumption, we have that

$$\frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\Big(e^{\alpha(x,x^*)\sum_{i=1}^{p}[L_i(x,u^*,v^*,\lambda^*,\bar{t},\bar{s})-L_i(x^*,u^*,v^*,\lambda^*,\bar{t},\bar{s})]} - 1\Big) \geqq 0.$$

We need to consider two cases: $\alpha(x,x^*) > 0$ and $\alpha(x,x^*) < 0$. If we assume that $\alpha(x,x^*) > 0$ and recall that $\gamma(x,x^*) > 0$, then the above inequality becomes

$$e^{\alpha(x,x^*)\sum_{i=1}^{p}[L_i(x,u^*,v^*,\lambda^*,\bar{t},\bar{s})-L_i(x^*,u^*,v^*,\lambda^*,\bar{t},\bar{s})]} \geqq 1,$$

which implies that

$$\sum_{i=1}^{p} L_i(x,u^*,v^*,\lambda^*,\bar{t},\bar{s}) \geqq \sum_{i=1}^{p} L_i(x^*,u^*,v^*,\lambda^*,\bar{t},\bar{s}).$$

Because $x^* \in \mathbb{F}$, $t^m \in \hat{T}_{j_m}(x^*), m \in \underline{\nu_0}$, and $\lambda_i^* = \varphi_i(x^*)$, $i \in \underline{p}$, the right-hand side of the above inequality is equal to zero, and hence we have $L(x,u^*,v^*,\lambda^*,\bar{t},\bar{s}) \geqq 0$. Next, as $x \in \mathbb{F}$, and $v_m^* > 0$, $m \in \underline{\nu_0}$, this inequality simplifies to

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda_i^* g_i(x)] \geqq 0. \tag{7}$$

Since $u^* > 0$, the above inequality implies that

$$\big(f_1(x) - \lambda_1^* g_1(x), \ldots, f_p(x) - \lambda_p^* g_p(x)\big) \nleq (0,\ldots,0),$$

which in turn implies that

$$\Big(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\Big) \nleq (\lambda_1^*, \ldots, \lambda_p^*) = \varphi(x^*).$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an efficient solution of (P). On the other hand, we arrive at the same conclusion if we assume that $\alpha(x,x^*) < 0$.

Remark. We observe that the proof for Theorem 3.1 can be achieved using the method of contradictions as well.

Theorem 3.2 *Let $x^* \in \mathbb{F}$, $\lambda^* = \varphi(x^*)$, the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot,t)$ and $H_k(\cdot,s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$. Assume further that there exist $u^* \in U$ and integers ν_0 and ν with $0 \leqq \nu_0 \leqq \nu \leqq n+1$ such that there exist ν_0 indices j_m with $1 \leqq j_m \leqq q$ together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m with*

$1 \leqq k_m \leqq r$ together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$ such that for each critical direction $z \in \mathbb{R}^n$, and (3.1) and (3.2) hold.

In addition, assume that any one of the following four sets of hypotheses is satisfied:

(a) (i) $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is higher order exponential type hybrid* $(\alpha, \beta, \bar{\gamma}, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-pseudoinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is higher order exponential type hybrid* $(\alpha, \beta, \hat{\gamma}, \omega(\cdot,\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-quasiinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is higher order exponential type hybrid* $(\alpha, \beta, \breve{\gamma}, \varpi(\cdot,\cdot,\cdot), \breve{\rho}, \eta, \theta)$*-quasiinvex at* x^*;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geqq 0$ *for all* $x \in \mathbb{F}$;

(b) (i) $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly higher order exponential type hybrid* $(\alpha, \beta, \bar{\gamma}, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is higher order exponential type hybrid* $(\alpha, \beta, \hat{\gamma}, \omega(\cdot,\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-quasiinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is order exponential type hybrid* $(\alpha, \beta, \breve{\gamma}, \varpi(\cdot,\cdot,\cdot), \breve{\rho}, \eta, \theta)$*-quasiinvex at* x^*;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(c) (i) $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly higher order exponential type hybrid* $(\alpha, \beta, \bar{\gamma}, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is strictly higher order exponential type hybrid* $(\alpha, \beta, \hat{\gamma}, \omega(\cdot,\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-pseudoinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is higher order exponential type hybrid* $(\alpha, \beta, \breve{\gamma}, \varpi(\cdot, \cdot$
cdot$), \breve{\rho}, \eta, \theta)$*-quasiinvex at* x^*;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geqq 0$ *for all* $x \in \mathbb{F}$;

(d) (i) $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly higher order exponential type* $(\alpha, \beta, \bar{\gamma}, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is higher order exponential type hybrid* $(\alpha, \beta, \hat{\gamma}, \omega(\cdot,\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-quasiinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is strictly higher order exponential type* $(\alpha, \beta, \breve{\gamma}, \varpi(\cdot,\cdot,\cdot), \breve{\rho}, \eta, \theta)$*-pseudoinvex at* x^*;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geqq 0$ *for all* $x \in \mathbb{F}$.

Then x^* *is an efficient solution to (P).*

(a): Let x be an arbitrary feasible solution to (P). Since $G_{j_m}(x,t^m) \leqq 0 = G_{j_m}(x^*,t^m)$, it follows that

$$\sum_{m=1}^{\nu_0} v_m^* \pi_m(x,x^*) G_{j_m}(x,t^m) \leqq \sum_{m=1}^{\nu_0} v_m^* \pi_m(x,x^*) G_{j_m}(x^*,t^m),$$

and so

$$\frac{1}{\alpha(x,x^*)} \hat{\gamma}(x,x^*) \Big(e^{\alpha(x,x^*) \sum_{m=1}^{\nu_0} \pi_m(x,x^*)[v_m^* G_{j_m}(x,t^m) - v_m^* G_{j_m}(x^*,t^m)]} - 1 \Big) \leqq 0$$

by using $\alpha(x,x^*) \neq 0$ and $\hat{\gamma}(x,x^*) \geqq 0$. In light of (ii), this inequality implies that

$$\begin{aligned} &\frac{1}{\beta(x,x^*)} \Big(\Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla_z \omega_{j_m}(x^*,t^m,z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \Big\rangle \\ &+ \Big\langle \sum_{m=1}^{\nu_0} v_m^* \omega_{j_m}(x^*,t^m,z), e^{\beta(x,x^*)} - 1 \Big\rangle \\ &- \Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla_z \omega_{j_m}(x^*,t^m,z), e^{\beta(x,x^*)z} - \mathbf{1} \Big\rangle \Big) \\ &\leqq -\hat{\rho}(x,x^*) \|\theta(x,x^*)\|^2. \end{aligned} \tag{8}$$

Similarly, assumptions in (iii) lead to the following inequality:

$$\begin{aligned} &\frac{1}{\beta(x,x^*)} \Big(\Big\langle \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla_z \varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \Big\rangle \\ &+ \Big\langle \sum_{m=\nu_0+1}^{\nu} v_m^* \varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)} - 1 \Big\rangle \\ &- \Big\langle \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla_z \varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)z} - \mathbf{1} \Big\rangle \\ &\leqq -\check{\rho}(x,x^*) \|\theta(x,x^*)\|^2. \end{aligned} \tag{9}$$

Now combining (3.1), (3.2), (3.8), and (3.9), and using (iv), we obtain

$$\begin{aligned} &\frac{1}{\beta(x,x^*)} \Big(\Big\langle \sum_{i=1}^{p} u_i^* [\nabla_z h_i(x^*,z) - \lambda_i^* \nabla_z \kappa_i(x^*,z)], e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \Big\rangle \\ &+ \Big\langle \sum_{i=1}^{p} u_i^* [\nabla_z h_i(x^*,z) - \lambda_i^* \nabla_z \kappa_i(x^*,z)], e^{\beta(x,x^*)} - 1 \Big\rangle \\ &- \Big\langle \sum_{i=1}^{p} u_i^* [\nabla_z h_i(x^*,z) - \lambda_i^* \nabla_z \kappa_i(x^*,z)], e^{\beta(x,x^*)z} - \mathbf{1} \Big\rangle \Big) \\ &\geqq -\bar{\rho}(x,x^*) \|\theta(x,x^*)\|^2, \end{aligned}$$

which in view of (i) implies that

$$\frac{1}{\alpha(x,x^*)} \bar{\gamma}(x,x^*) \Big(e^{\alpha(x,x^*) \sum_{i=1}^{p} u_i^* \{f_i(x) - \lambda_i^* g_i(x) - [f_i(x^*) - \lambda_i^* g_i(x^*)]\}} - 1 \Big) \geqq 0.$$

Since $\bar{\gamma}(x,x^*) > 0$ and $\varphi(x^*) = \lambda^*$, this inequality implies that

$$\sum_{i=1}^{p} u_i^* [f_i(x) - \lambda_i^* g_i(x)] \geqq 0.$$

Based on the proof of Theorem 3.1, we conclude that x^* is an efficient solution to (P).

(b) - (e): The proofs are similar to that of part (a).

We also observe that Theorem 3.2 generalizes a result of Zalmai ([53], Theorem 4.1) on the sufficient efficiency conditions.

Theorem 3.3 *Let $x^* \in \mathbb{F}$, $\lambda^* = \varphi(x^*)$, the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot,t)$, and $H_k(\cdot,s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (3.1) holds. Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:*

(a) *(i)* $\big(\mathcal{E}_1(\cdot,\lambda^*,u^*),\dots,\mathcal{E}_p(\cdot,\lambda^*,u^*)\big)$ *is* $HA(\alpha,\beta,\bar{\gamma},\xi,\eta,\bar{\rho},\theta)$*-V-pseudoinvex at* x^* *and* $\bar{\gamma}(x,x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot,t^1),\dots,v_{\nu_0}^* G_{j_{\nu_0}}(\cdot,t^{\nu_0})\big)$ *is* $HA(\alpha,\beta,\hat{\gamma},\pi,\eta,\hat{\rho},\theta)$*-V-quasiinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}),\dots,v_\nu^* H_{k_\nu}(\cdot,s^\nu)\big)$ *is* $HA(\alpha,\beta,\breve{\gamma},\delta,\eta,\breve{\rho},\theta)$*-V-quasiinvex at* x^*;

(iv) $\rho(x,x^*) + \hat{\rho}(x,x^*) + \breve{\rho}(x,x^*) \geqq 0$ *for all* $x \in \mathbb{F}$;

(b) *(i)* $\big(\mathcal{E}_1(\cdot,\lambda^*,u^*),\dots,\mathcal{E}_p(\cdot,\lambda^*,u^*)\big)$ *is prestrictly* $HA(\alpha,\beta,\bar{\gamma},\xi,\eta,\bar{\rho},\theta)$*-V-quasiinvex at* x^* *and* $\bar{\gamma}(x,x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot,t^1),\dots,v_{\nu_0}^* G_{j_{\nu_0}}(\cdot,t^{\nu_0})\big)$ *is* $HA(\alpha,\beta,\hat{\gamma},\pi,\eta,\hat{\rho},\theta)$*-V-quasiinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}),\dots,v_\nu^* H_{k_\nu}(\cdot,s^\nu)\big)$ *is* $HA(\alpha,\beta,\breve{\gamma},\delta,\eta,\breve{\rho},\theta)$*-V-quasiinvex at* x^*;

(iv) $\bar{\rho}(x,x^*) + \hat{\rho}(x,x^*) + \breve{\rho}(x,x^*) > 0$ *for all* $x \in \mathbb{F}$;

(c) *(i)* $\big(\mathcal{E}_1(\cdot,\lambda^*,u^*),\dots,\mathcal{E}_p(\cdot,\lambda^*,u^*)\big)$ *is prestrictly* $HA(\alpha,\beta,\bar{\gamma},\xi,\eta,\bar{\rho},\theta)$*-V-quasiinvex at* x^* *and* $\bar{\gamma}(x,x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot,t^1),\dots,v_{\nu_0}^* G_{j_{\nu_0}}(\cdot,t^{\nu_0})\big)$ *is strictly* $HA(\alpha,\beta,\hat{\gamma},\pi,\eta,\hat{\rho},\theta)$*-V-pseudoinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}),\dots,v_\nu^* H_{k_\nu}(\cdot,s^\nu)\big)$ *is* $HA(\alpha,\beta,\breve{\gamma},\delta,\eta,\breve{\rho},\theta)$*-V-quasiinvex at* x^*;

(iv) $\bar{\rho}(x,x^*) + \hat{\rho}(x,x^*) + \breve{\rho}(x,x^*) \geqq 0$ *for all* $x \in \mathbb{F}$;

(d) *(i)* $\big(\mathcal{E}_1(\cdot,\lambda^*,u^*),\dots,\mathcal{E}_p(\cdot,\lambda^*,u^*)\big)$ *is prestrictly* $HA(\alpha,\beta,\bar{\gamma},\xi,\eta,\bar{\rho},\theta)$*-V-quasiinvex at* x^* *and* $\bar{\gamma}(x,x^*) > 0$ *for all* $x \in \mathbb{F}$;

(ii) $\big(v_1^* G_{j_1}(\cdot,t^1),\dots,v_{\nu_0}^* G_{j_{\nu_0}}(\cdot,t^{\nu_0})\big)$ *is* $HA(\alpha,\beta,\hat{\gamma},\pi,\eta,\hat{\rho},\theta)$*-V-quasiinvex at* x^*;

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}),\dots,v_\nu^* H_{k_\nu}(\cdot,s^\nu)\big)$ *is strictly* $HA(\alpha,\beta,\breve{\gamma},\delta,\eta,\breve{\rho},\theta)$*-V-pseudoinvex at* x^*;

(iv) $\bar{\rho}(x,x^*) + \hat{\rho}(x,x^*) + \breve{\rho}(x,x^*) \geqq 0$ *for all* $x \in \mathbb{F}$.

Then x^* *is an efficient solution of (P).*

Next, we present the dual problem (DI) to primal problem (P) based on the efficiency conditions for (P). We consider a parametric duality problem to (P) based on the parametric efficiency conditions for (P) as follows:

(DI) Maximize $\lambda = (\lambda_1, \cdots, \lambda_p)$

subject to

$$\sum_{i=1}^{p} u_i[\nabla_z h_i(y,z) - \lambda \nabla_z \kappa_i(y,z)] + \sum_{m=1}^{\nu_0} v_m \nabla_z \omega_{j_m}(y,t^m,z) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla_z \varpi_{k_m}(y,s^m,z) = 0, \tag{10}$$

$$\sum_{i=1}^{p} u_i[f_i(y) - \lambda_i g_i(y)] + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y,s^m) \geqq 0. \tag{11}$$

It can be shown that (DI) is a dual problem to (P) by applying higher order exponential type hybrid invexity assumptions. Let x and y be arbitrary feasible solutions to (P) and (DI), respectively. Assume that the function $L(\cdot,u,v,\lambda,\bar{t},\bar{s}) : X \to \mathbb{R}^p$ defined by

$$L(\zeta,u,v,\lambda) = \Big(L_1(\zeta,u,v,\lambda,\bar{t},\bar{s}), \cdots, L_p(\zeta,u,v,\lambda,\bar{t},\bar{s})\Big)$$

is higher order exponential type hybrid (α, β, γ, η, $h(\cdot,\cdot)$, $\kappa(\cdot,\cdot)$, $\omega(\cdot,\cdot,\cdot)$, $\varpi(\cdot,\cdot,\cdot)$, ρ, θ)-pseudoinvex at y for $\gamma(x,y) > 0$, where

$$L_i(\zeta,u,v,\lambda,\bar{t},\bar{s}) = u_i\Big[f_i(\zeta) - \lambda_i g_i(\zeta) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(\zeta,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\zeta,s^m)\Big],\ i \in \underline{p}.$$

Then from the pseudoinvexity assumption and (3.10), it follows that

$$\frac{1}{\alpha(x,y)}\gamma(x,y)\Big(e^{\alpha(x,y)\sum_{i=1}^{p}[L_i(x,u,v,\lambda,\bar{t},\bar{s})-L_i(y,u,v,\lambda,\bar{t},\bar{s})} - 1\Big) \geqq 0.$$

If we assume that $\alpha(x,y) > 0$ (while we arrive at the same conclusion for $\alpha(x,y) < 0$) and $\gamma(x,y) > 0$, then we have

$$e^{\alpha(x,y)\sum_{i=1}^{p}\left[L_i(x,u,v,\lambda,\bar{t},\bar{s})-L_i(y,u,v,\lambda,\bar{t},\bar{s})\right.} \geqq 1.$$

This implies

$$\sum_{i=1}^{p} L_i(x,u,v,\lambda,\bar{t},\bar{s}) \geqq \sum_{i=1}^{p}\left[L_i(y,u,v,\lambda,\bar{t},\bar{s}) \geqq 0.\right.$$

Since $x \in \mathbb{F}$ and $v_m > 0,\ m \in \underline{\nu_0}$, the above inequality reduces to

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda_i g_i(x)] \geqq 0. \tag{12}$$

Since $u > 0$, $i \in \underline{p}$, it further follows that

$$\big(f_1(x) - \lambda_1 g_1(x), \ldots, f_p(x) - \lambda_p g_p(x)\big) \nleq (0,\ldots,0),$$

which in turn implies that

$$\varphi(x) = \Big(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\Big) \nleq (\lambda_1, \ldots, \lambda_p) = \lambda.$$

This results in $\varphi(x) \nleq \lambda$, that is, (DI) is a dual problem to (P).
Furthermore, the dual problem (DI) generalizes most of duality models, especially to the context of semiinfinite multiobjective fractional programming problems.

4 Concluding Remarks

We established several families of sufficient efficiency results under various higher order exponential type hybrid $(\alpha, \beta, \gamma, \eta, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot,\cdot), \varpi(\cdot,\cdot,\cdot), \rho, \theta)$-invexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions based on a certain type of partitioning scheme. This hybrid invexity framework encompasses most of the generalized invexity notions even by specializing the function $h(\cdot,\cdot)$ to more frequently used functions such as f for other application purposes. In light of this hybrid invexity notion, other notions of generalized univexities can be applied to duality models as well. On the other hand, one can apply a more generalized partitioning scheme as follows: let ν_0 and ν be integers with $1 \leqq \nu_0 \leqq \nu \leqq n+1$, and let $\{J_0, J_1, \ldots, J_M\}$ and $\{K_0, K_1, \ldots, K_M\}$ be partitions of the sets $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively; thus, $J_i \subseteq \underline{\nu_0}$ for each $i \in \underline{M} \cup \{0\}$, $J_i \cap J_j = \emptyset$ for each $i, j \in \underline{M} \cup \{0\}$ with $i \neq j$, and $\cup_{i=0}^{M} J_i = \underline{\nu_0}$. Clearly, similar properties hold for $\{K_0, K_1, \ldots, K_M\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \emptyset$ or $K_i = \emptyset$ for $i > \min\{m_1, m_2\}$.

Furthermore, based on the new class of multitime multiobjective variational problems for duality theorems (introduced and investigated by Pitea and Postolache [35 - 37]), we observe that our results can be applied to this class of multitime multiobjective variational problems. It seems that the work of Pitea and Postolache [35 - 37] on multitime multiobjective variational problems offers greater opportunities for further future research endeavors based on other aspects of generalized invexities. On the other hand, there is also a greater scope for further research advances for applying the Mond-Smart generalized invexities for multiobjective control problems relating to duality as well as other aspects of the generalized invexities. Zhian and Qingkai [55] investigated duality models for a class of multiobjective control problems based on the generalized invexity.

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