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A short note on fuzzy residual sets and fuzzy functions

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ABSTRACT. In this paper several characterizations of fuzzy residual sets are established and the criteria for preservation of the fuzzy residuality of fuzzy sets under fuzzy continuous and fuzzy open functions between fuzzy topological spaces, are obtained.

1. Introduction:

The notion of a fuzzy set introduced by L.A.Zadeh [15] in 1965, has caused great interest among both pure and applied mathematicians. Since then, the notion of fuzziness has been applied for the study in all branches of mathematics. General topology was one of the first branches of pure mathematics to which fuzzy sets have been applied systematically. In 1968, C.L.Chang [3] introduced fuzzy topological spaces by using fuzzy sets. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since the early eighties, the intensity of research in the area of fuzzy topology has increased sharply.

In classical topology, Baire space named in honor of Rene Louis Baire, was first introduced in Bourbaki's [2] Topologie Generale Chapter IX. Baire spaces have been studied extensively in classical topology in [7] and [9]. Z. Frolik [6], T. Neubrunn [9] and Jozef Dobos et.al., [4] applied several types of continuous functions that preserve Baireness in the context of images and pre-images in topological spaces. The notion of fuzzy continuity introduced by C. L. Chang [3] has proved to be of fundamental importance in the realm of fuzzy topology [5, 14]. The concept of Baireness in fuzzy setting was introduced and studied by G.Thangaraj and S.Anjalmose in [13]. The purpose of this paper is to obtain several characterizations of fuzzy residual sets and find the criteria

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for preservation of the fuzzy residuality of fuzzy sets, by the preimages of fuzzy continuous functions and by the images of fuzzy open functions between fuzzy topological spaces.

2. Preliminaries:

In order to make the exposition self-contained, some basic notions and results used in the sequel are given. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I, the unit interval [0,1]. A fuzzy set λ in X is a function from X into I. The null set 0 is the function from X into I which assumes only the value 0 and the whole fuzzy set 1 is the function from X into I which takes 1 only.

Definition 2.1. [1] Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior and the closure of λ are defined as follows:

(i) $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$

(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1-\mu \in T\}$.

Lemma 2.1. [1] Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1-\lambda)$ and $1 - int(\lambda) = cl(1-\lambda)$.

Definition 2.2. [11] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is., $int[cl(\lambda)] = 0$, in (X, T).

Definition 2.3. [10] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy somewhere dense set if $int[cl(\lambda)] \neq 0$ for any fuzzy set $\lambda(\neq 0)$ in (X, T).

Definition 2.4. [11] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.5. [13] Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1-\lambda$ is called a fuzzy residual set in (X, T).

Definition 2.6. [13] Let (X,T) be a fuzzy topological space. Then (X, T) is called a fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T).

Theorem 2.1. [13] *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) (X, T) is a fuzzy Baire space.

(2) $int(\lambda) = 0$ for every fuzzy first category set λ in (X, T).

(3) $cl(\mu) = 1$ for every fuzzy residual set μ in (X, T).

Definition 2.7. [12] A fuzzy topological space (X, T) is called a fuzzy resolvable space if there exists a fuzzy dense set λ in (X, T) such that $1-\lambda$ is also a fuzzy dense set in (X, T). Otherwise (X, T) is called a fuzzy irresolvable space.

Definition 2.8. [3] A fuzzy topological space (X, T) is called a fuzzy compact space, if $\vee_{i \in I}(A_i) = 1$, where $A_i \in T$, then there are finitely many indices $i_1, i_2, ...i_n \in I$ such that $\vee_{j=1}^{N} A_{i_j} = 1$.

Definition 2.9. [8] Let λ be a fuzzy set in a fuzzy topological space (X, T). The fuzzy boundary of λ is defined as $Bd(\lambda) = cl(\lambda) \wedge cl(1-\lambda)$.

3. Fuzzy Residual sets:

Definition 3.1. Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a fuzzy residual set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where the fuzzy sets $(\lambda_i)'s$ are such that $cl[int(\lambda_i)] = 1$ in (X, T).

Example 3.1: Let $X = \{a, b, c\}$. Consider the fuzzy sets λ, μ α, β, γ, δ, η, θ and τ defined on X as follows:

$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 1; \lambda(b) = 0.2; \lambda(c) = 0.7.$

$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.7; \mu(b) = 0.4; \mu(c) = 1.$

$\alpha : X \to [0, 1]$ defined as $\alpha(a) = 0.7; \alpha(b) = 0.3; \alpha(c) = 0.7;$

$\beta : X \to [0, 1]$ defined as $\beta(a) = 1; \beta(b) = 0.5, \beta(c) = 0.8;$

$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.8; \gamma(b) = 0.6; \gamma(c) = 0.9;$

$\delta : X \to [0, 1]$ defined as $\delta(a) = 1; \delta(b) = 0.9; \delta(c) = 0.8;$

$\eta, : X \to [0, 1]$ defined as $\eta(a) = 0.9; \eta(b) = 0.7; \eta(c) = 0.8;$

$\theta : X \to [0, 1]$ defined as $\theta(a) = 0.8; \theta(b) = 0.9; \theta(c) = 0.7$ and

$\tau : X \to [0, 1]$ defined as $\tau(a) = 0.7; \tau(b) = 0.4; \tau(c) = 0.7.$

Clearly, $T = \{0, \lambda, \mu, (\lambda \vee \mu), (\lambda \wedge \mu), 1\}$ is a fuzzy topology on X. On computations, one can see that $cl[int(\lambda)] = 1$; $cl[int(\mu)] = 1$; $cl[int(\lambda \vee \mu)] = 1$; $cl[int(\lambda \wedge \mu)] = 1$; $cl[int(\alpha)] = 1$; $cl[int(\beta)] = 1$; $cl[int(\gamma)] = 1$; $cl[int(\delta)] = 1$; $cl[int(\eta)] = 1$; $cl[int(\theta)] = 1$ and $(\lambda \wedge \mu) = (\lambda) \wedge (\mu) \wedge (\lambda \vee \mu) \wedge (\alpha) \wedge (\beta) \wedge (\gamma) \wedge (\delta) \wedge (\eta) \wedge (\theta)$, $\tau = (\mu) \wedge (\lambda \vee \mu) \wedge (\beta) \wedge (\gamma) \wedge (\delta) \wedge (\eta) \wedge (\theta)$ are fuzzy residual sets in (X, T). It should be noted that a fuzzy residual set need not be fuzzy open, since τ is not fuzzy open in (X, T).

Proposition 3.1: If λ is a fuzzy G_δ set in a fuzzy topological space (X, T) such that $cl(\lambda) = 1$, then λ is a fuzzy residual set in (X, T).

Proof : Let λ be a fuzzy G_δ set in (X, T). Then $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$. Now $cl(\lambda) = cl[\wedge_{i=1}^{\infty}(\lambda_i)]$ in (X, T). But $cl[\wedge_{i=1}^{\infty}(\lambda_i)] \leq \wedge_{i=1}^{\infty} cl(\lambda_i)$ in (X, T). Then $cl(\lambda) \leq \wedge_{i=1}^{\infty} cl(\lambda_i)$ in (X, T). Since λ is a fuzzy dense set in (X, T), $cl(\lambda) = 1$ and hence $1 \leq \wedge_{i=1}^{\infty} cl(\lambda_i)$. That is., $\wedge_{i=1}^{\infty} cl(\lambda_i) = 1$ in (X, T). This implies that $cl(\lambda_i) = 1$ in (X, T) and hence $cl[int(\lambda_i)] = cl(\lambda_i) = 1$ in (X, T). Thus $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $cl[int(\lambda_i)] = 1$, implies that λ is a fuzzy residual set in (X, T).

Proposition 3.2: If $cl(\lambda)$ is a fuzzy first category set in a fuzzy Baire space (X, T), then λ is a fuzzy nowhere dense set in (X, T).

Proof : Let $cl(\lambda)$ be a fuzzy first category set in (X, T). Since (X, T) is a fuzzy Baire space, by theorem 2.1, $int[cl(\lambda)] = 0$ in (X, T). Therefore λ is a fuzzy nowhere dense set in (X, T).

Proposition 3.3: If $int(\mu)$ is a fuzzy residual set in a fuzzy Baire space (X, T), then $1 - \mu$ is a fuzzy nowhere dense set in (X, T).

Proof : Let $int(\mu)$ be a fuzzy residual set in (X, T). Since (X, T) is a fuzzy Baire space, by theorem 2.1, $cl[int(\mu)] = 1$ in (X, T). This implies that $1 - cl[int(\mu)] = 0$ and hence $int[cl(1 - \mu)] = 0$, in (X, T). This implies that $1 - \mu$ is a fuzzy nowhere dense set in (X, T).

Proposition 3.4: If $[int(\mu_i)]'s$ $(i = 1$ to $\infty)$ are fuzzy residual sets in a fuzzy Baire space (X, T), then $cl[\wedge_{i=1}^{\infty}(\mu_i)] = 1$ in (X, T).

Proof : Let $[int(\mu_i)]'s$ $(i = 1$ to $\infty)$ be fuzzy residual sets in (X, T). Since (X, T) is a fuzzy Baire space, by proposition 3.3, $(1-\mu_i)'s$ are fuzzy nowhere dense sets in (X,T) and hence $int[\vee_{i=1}^{\infty}(1-\mu_i)] = 0$ in (X,T). This implies that $int[1-\wedge_{i=1}^{\infty}(\mu_i)] = 0$. Then, $1 - cl[\wedge_{i=1}^{\infty}(\mu_i)] = 0$ in (X,T) and thus $cl[\wedge_{i=1}^{\infty}(\mu_i)] = 1$ in (X,T).

Definition 3.2: A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy semi - G_δ set in (X,T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $(\lambda_i)'s$ are fuzzy semi-open sets in (X,T).

Definition 3.3: A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy semi - F_σ set in (X,T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $(\lambda_i)'s$ are fuzzy semi - closed sets in (X,T).

Remark 3.1: If λ is a fuzzy semi - G_δ set in a fuzzy topological space (X,T) then $1-\lambda$ is a fuzzy semi - F_σ set in (X,T). For, $\lambda = \wedge_{i=1}^{\infty}\lambda_i$, where $(\lambda_i)'s$ are fuzzy semi - open sets in (X,T) implies that $1-\lambda = \vee_{i=1}^{\infty}(1-\lambda_i)$ where $(1-\lambda_i)'s$ are fuzzy semi - closed sets in (X,T).

Proposition 3.5: If λ is a fuzzy residual set in a fuzzy topological space (X,T), then λ is a fuzzy semi - G_δ set.

Proof: Let λ be a fuzzy residual set in a fuzzy topological space (X,T). Then, $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where the fuzzy sets $(\lambda_i)'s$ are such that $cl[int(\lambda_i)] = 1$ in (X,T). Now $cl[int(\lambda_i)] = 1$ in (X,T), implies that $\lambda_i \leq cl[int(\lambda_i)]$. Then $(\lambda_i)'s$ are fuzzy semi - open sets in (X,T) and thus $\wedge_{i=1}^{\infty}(\lambda_i)$ is a fuzzy semi - G_δ set in (X,T).

Proposition 3.6: If λ is a fuzzy first category set in a fuzzy topological space (X,T), then λ is a fuzzy semi - F_σ set in (X,T).

Proof : Let λ be a fuzzy first category set in (X,T). Then $1-\lambda$ is a fuzzy residual set in (X,T) and hence, by proposition 3.5, $1-\lambda$ is a fuzzy semi - G_δ set in (X,T). Therefore λ is a fuzzy semi - F_σ set in (X,T).

Proposition 3.7: If $int(\mu)$ and $int(1-\mu)$ are fuzzy residual sets in a fuzzy Baire space (X,T), then $bd(\mu) = 1$ in (X,T).

Proof: Let $int(\mu)$ and $int(1-\mu)$ be fuzzy residual sets in (X,T). Then by Proposition 3.3, $1-\mu$ and $1-(1-\mu)$ are fuzzy nowhere dense sets in (X,T). Then $int[cl(1-\mu)] = 0$ and $int[cl(\mu)] = 0$ in (X,T). But $int(1-\mu) \leq int[cl(1-\mu)]$ and $int(\mu) \leq int[cl(\mu)]$ implies that $int(1-\mu) = 0$ and $int(\mu) = 0$. Hence $1 - cl(\mu) = 0$ and $1 - int(\mu) = 1$ in (X,T). Then, $cl(\mu) = 1$ and $cl(1-\mu) = 1$ in (X,T). Now the fuzzy boundary of μ is $bd(\mu) = cl(\mu) \wedge cl(1-\mu) = 1 \wedge 1 = 1$. Thus $bd(\mu) = 1$ in (X,T).

Proposition 3.8: If $int(\mu)$ and $int(1-\mu)$ are fuzzy residual sets in a fuzzy Baire space (X,T), then (X,T) is a fuzzy resovable space.

Proof: Let $int(\mu)$ and $int(1-\mu)$ be fuzzy residual sets in (X,T). Since (X,T) is a fuzzy Baire space, by theorem 2.1, $cl[int(\mu)] = 1$ and $cl[int(1-\mu)] = 1$ in (X,T). Now $cl[int(\mu)] \leq cl(\mu)$ and $cl[int(1-\mu)] \leq cl(1-\mu)$ implies that $1 \leq cl(\mu)$ and $1 \leq cl(1-\mu)$. That is., $cl(\mu) = 1$ and $cl(1-\mu) = 1$ in (X,T) and hence (X,T) is a fuzzy resolvable space.

4. Fuzzy Baire spaces and fuzzy continuous functions:

If $f : (X,T) \to (Y,S)$ is a fuzzy continuous function from (X,T) into (Y,S) and δ is a fuzzy residual set in (Y,S), then $f^{-1}(\delta)$ need not be a fuzzy residual set in (X,T).

For, consider the following example:

Example 4.1:

Let $X = \{a,b\}$. The fuzzy sets λ, μ, α, β and γ are defined on X as follows:

$\lambda : X \to [0,1]$ is defined as $\lambda(a) = 0.5; \lambda(b) = 0.7;$

$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.8; \mu(b) = 0.4;$

$\alpha : X \to [0,1]$ is defined as $\alpha(a) = 0.2; \alpha(b) = 0.6;$

$\beta : X \to [0,1]$ is defined as $\beta(a) = 0.8; \beta(b) = 0.9$ and

$\gamma : X \to [0,1]$ is defined as $\gamma(a) = 0.9; \gamma(b) = 0.8.$

Then $T = \{0, \lambda, \mu, \alpha, (\lambda \vee \mu), (\mu \vee \alpha), (\lambda \wedge \mu), (\mu \wedge \alpha), [\alpha \vee (\lambda \wedge \mu)], 1\}$ and $S = \{0, \lambda, \mu, (\lambda \vee \mu), (\lambda \wedge \mu), 1\}$ are fuzzy topologies on X. On computations, one can find that $cl[int(\lambda)] = 1$, $cl[int(\lambda \vee \mu)] = 1$, $cl[int(\beta)] = 1$, $cl[int(\gamma)] = 1$ in (X, T) and $(\lambda) \wedge (\lambda \vee \mu) \wedge (\beta) \wedge (\gamma) = \lambda$ implies that λ is the fuzzy residual set in (X, T). Also,$cl[int(\lambda)] = 1$; $cl[int(\mu)] = 1$; $cl[int(\lambda \vee \mu)] = 1$; $cl[int(1 - \alpha)] = 1$; $cl[int(1 - [\mu \wedge \alpha])] = 1$; $cl[int(\beta)] = 1$; $cl[int(\gamma)] = 1$; in (X, S) and $\lambda \wedge \mu = (\lambda) \wedge (\mu) \wedge (\lambda \vee \mu) \wedge (1 - \alpha) \wedge (1 - [\mu \wedge \alpha]) \wedge (\beta) \wedge (\gamma)$ implies that $\lambda \wedge \mu$ is a fuzzy residual set in (X, S). Now, define a function $f : (X, T) \to (X, S)$ by $f(a) = a$ and $f(b) = b$. Clearly f is a fuzzy continuous funtion from (X, T) into (X, S). Now, $f^{-1}(\lambda \wedge \mu) = \lambda \wedge \mu$ and $\lambda \wedge \mu$ is not a fuzzy residual set in (X, T), shows that $f^{-1}(\lambda \wedge \mu)$ is not a fuzzy residual set in (X, T).

If $f : (X, T) \to (Y, S)$ is a fuzzy open function from (X, T) into (Y, S) and η is a fuzzy residual set in (X, T), then $f(\eta)$ need not be a fuzzy residual set in (Y, S).

For, consider the following example:

Example 4.2:

Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, δ are defined on X as follows:

$\lambda : X \to [0,1]$ is defined as $\lambda(a) = 0.4; \lambda(b) = 0.6; \lambda(c) = 0.5;$

$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.7; \mu(b) = 0.4; \mu(c) = 0.4;$

$\delta : X \to [0,1]$ is defined as $\delta(a) = 0.3; \delta(b) = 0.6; \delta(c) = 0.6;$

Then $T = \{0, \lambda, \mu, (\lambda \vee \mu), (\lambda \wedge \mu), 1\}$ and $S = \{0, \lambda, \mu, \delta, (\lambda \vee \mu), (\lambda \vee \delta), (\mu \vee \delta), (\lambda \wedge \mu), (\lambda \wedge \delta), (\mu \wedge \delta), 1\}$ are fuzzy topologies on X. Define a function $f : (X, T) \to (X, S)$ by $f(a) = a, f(b) = b, f(c) = c$. Clearly f is a fuzzy open function from (X, T) into (X, S). On computations, we see that $cl[int(\mu)] = 1$, $cl[int(\lambda \vee \mu)] = 1$ and hence $\mu = (\mu) \wedge (\lambda \vee \mu)$ is a fuzzy residual set in (X, T). On computation $f(\mu) = \mu$ and μ is not a fuzzy residual set in (X, S).

Proposition 4.1: If $f : (X, T) \to (Y, S)$ is a fuzzy continuous function from a fuzzy topological space (X, T) onto a fuzzy topological space (Y, S) such that $f(\mu)$ is a fuzzy somewhere dense set in (Y, S) for each non-zero fuzzy open set μ in (X, T), and if λ is a fuzzy residual set in (Y, S), then $f^{-1}(\lambda)$ is a fuzzy residual set in (X, T).

Proof: Let λ be a fuzzy residual set in (Y, S). Then $1 - \lambda$ is a fuzzy first category set in (Y, S) and hence $1 - \lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (Y, S). Now, $f^{-1}(1 - \lambda) = f^{-1}[\vee_{i=1}^{\infty}(\lambda_i)] = \vee_{i=1}^{\infty}[f^{-1}(\lambda_i)]$ and $f^{-1}(1 - \lambda) = 1 - [f^{-1}(\lambda)]$, implies that $1 - f^{-1}(\lambda) = \vee_{i=1}^{\infty}[f^{-1}(\lambda_i)]$——-(1).

Consider the fuzzy set $\mu_i = int(f^{-1}[cl(\lambda_i)])$ in (X, T). Suppose that $\mu_i \neq 0$. Then μ_i is a non-zero fuzzy open set in (X, T). Now $\mu_i = int(f^{-1}[cl(\lambda_i)])$, implies that $f(\mu_i) = f[int(f^{-1}[cl(\lambda_i)])] \leq f(f^{-1}[cl(\lambda_i)]) = cl(\lambda_i)$. (since f is onto, $f(f^{-1}[cl(\lambda_i)]) = cl(\lambda_i)$). That is., $f(\mu_i) \leq cl(\lambda_i)$. Then we have $int(cl[f(\mu_i)]) \leq int(cl[cl(\lambda_i)]) = int(cl[\lambda_i])$——(2).

By hypothesis, $f(\mu_i)$ is a fuzzy somewhere dense set in (Y,S). That is., $int(cl[f(\mu_i)]) \neq 0$, and hence from (2), $int[cl(\lambda_i)] \neq 0$, a contradiction to (λ_i) being a fuzzy nowhere dense set (Y,S). Hence we must have $\mu_i = 0$ in (X,T), and hence $int(f^{-1}[cl(\lambda_i)]) = 0$, in (X,T). Since f is a fuzzy continuous function from (X,T) onto (Y,S), $cl[f^{-1}(\lambda_i)] \leq f^{-1}[cl(\lambda_i)]$ for the fuzzy set λ_i in (Y,S). Then, we have $int(cl[f^{-1}(\lambda_i)]) \leq int(f^{-1}[cl(\lambda_i)])$ and hence $int(cl[f^{-1}(\lambda_i)]) \leq 0$. That is., $int(cl[f^{-1}(\lambda_i)]) = 0$ in (X,T). Hence $f^{-1}(\lambda_i)$ is a fuzzy nowhere dense set in (X,T) and from (1), we have $1 - f^{-1}(\lambda)$ is a fuzzy first category set in (X,T). Therefore $f^{-1}(\lambda)$ is a fuzzy residual set in (X,T).

Proposition 4.2: If $f : (X,T) \rightarrow (Y,S)$ is a fuzzy open and one-to-one function from a fuzzy topological space (X,T) onto a fuzzy topological space (Y,S) such that $f^{-1}(\eta)$ is a fuzzy somewhere dense set in (X,T) for each non-zero fuzzy open set η in (Y,S), and if δ is a fuzzy residual set in (X,T) then $f(\delta)$ is a fuzzy residual set in (Y,S).

Proof: Let δ be a fuzzy residual set in (X,T). Then $1-\delta$ is a fuzzy first category set in (X,T) and hence $1-\delta = \vee_{i=1}^{\infty}(\delta_i)$, where $(\delta_i)'s$ are fuzzy nowhere dense sets in (X,T). Now, $(1-\delta) = \vee_{i=1}^{\infty}(\delta_i)$, implies that $f(1-\delta) = f(\vee_{i=1}^{\infty}(\delta i)) = \vee_{i=1}^{\infty}[f(\delta_i)]$. Since f is a one-to-one and onto function $f(1-\delta) = 1 - f(\delta)$, in (Y,S). Then, we have $1 - f(\delta) = \vee_{i=1}^{\infty} f(\delta_i)$——-(1).

Now consider the fuzzy set $\eta_i = int(f[cl(\delta_i)])$ in (Y,S). Suppose that $\eta_i \neq 0$. Then η_i is a non-zero fuzzy open set in (Y,S). Now $\eta_i = int(f[cl(\delta_i)])$ implies that $f^{-1}(\eta_i) = f^{-1}[int(f[cl(\delta_i)])] \leq f^{-1}(f[cl(\delta_i)]) = cl(\delta_i)$. (since f is one-to-one, $f^{-1}(f[cl(\delta_i)]) = cl(\delta_i)$). Then, we have $int(cl[f^{-1}(\eta_i)]) \leq int(cl[cl(\delta_i)]) = int[cl(\delta_i)]$——-(2).

By hypothesis, $f^{-1}(\eta_i)$ is a fuzzy somewhere dense set in (X,T). That is., $int(cl[f^{-1}(\eta_i)]) \neq 0$, in (X,T) and hence from (2), we have $int[cl(\delta_i)] \neq 0$, a contradiction to δ_i being a fuzzy nowhere dense set in (X,T). Hence we must have $\eta_i = 0$ in (Y,S). Thus, $int(f[cl(\delta_i)]) = 0$ in (Y,S)——-(3).

Now $\delta_i \leq cl(\delta_i)$ implies that $1-\delta_i \geq 1 - cl(\delta_i)$. Then we have $f(1-\delta_i) \geq f[1-cl(\delta_i)]$ and hence $f(1-\delta_i) \geq f[int(1-\delta_i)]$. Since f is an open function and $int(1-\delta_i)$ is an fuzzy open set in (X,T), then $f[int(1-\delta_i)]$ is an fuzzy open set in (Y,S) and hence $f[1-cl(\delta_i)]$ is an fuzzy open set in (Y,S). Since f is one-to-one and onto, $f[1-cl(\delta_i)] = 1 - f[cl(\delta_i)]$ and $1 - f[cl(\delta_i)]$ is an fuzzy open set in (Y,S), implies that $f[cl(\delta_i)]$ is a fuzzy closed set in (Y,S). Now $\delta_i \leq cl(\delta_i)$ implies that $f(\delta_i) \leq f[cl(\delta_i)]$ and hence $f[cl(\delta_i)]$ is a fuzzy closed set in (Y,S) such that $f(\delta_i) \leq f[cl(\delta_i)]$ and thus $cl[f(\delta_i)] \leq f[cl(\delta_i)]$. Then, $int(cl[f(\delta_i)]) \leq int(f[cl(\delta_i)])$ and from (3), $int(cl[f(\delta_i)]) \leq 0$. That is., $int(cl[f(\delta_i)]) = 0$, in (Y,S). Hence $f(\delta_i)$ is a fuzzy nowhere dense set in (Y,S) and hence from (1), $1 - f(\delta)$ is a fuzzy first category set in (Y,S). Therefore, $f(\delta)$ is a fuzzy residual set in (Y,S).

Proposition 4.3: If $f : (X,T) \rightarrow (Y,S)$ is a somewhat fuzzy continuous function from a fuzzy Baire space into a fuzzy topological space (Y,S) and λ is a fuzzy open set in (Y,S) then $f^{-1}(\lambda)$ is a fuzzy second category set in (X,T).

Proof: Let λ be a fuzzy open set in (Y,S). Since f is a somewhat fuzzy continuous function and $\lambda \in S$ implies that $int[f^{-1}(\lambda)] \neq 0$. We claim that $f^{-1}(\lambda)$ is a fuzzy second category set in (X,T). Assume the contrary. Suppose that $f^{-1}(\lambda)$ is a fuzzy first category set in (X,T). Since (X,T) is a fuzzy Baire space, by theorem 2.1, $int[f^{-1}(\lambda)] = 0$ a contradiction. Hence $f^{-1}(\lambda)$ must be a fuzzy second category set in (X,T).

Proposition 4.4: If $f : (X,T) \rightarrow (Y,S)$ is a somewhat fuzzy continuous function from a fuzzy Baire space (X,T) into a fuzzy topological space (Y,S) and if $\vee_{i=1}^{\infty}(\lambda_i) = 1$ in (Y,S), where $\lambda_i \in S$ then $\vee_{i=1}^{\infty} f^{-1}(\lambda_i) = 1$ in (X,T)

where $f^{-1}(\lambda_i)'s$ are fuzzy second category sets in (X,T).

Proof: Suppose that $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where $\lambda_i \in S$. Since f is somewhat fuzzy continuous from the fuzzy Baire space (X,T) into (Y,S) and $(\lambda_i)'s$ are fuzzy open sets in (Y,S), by proposition 4.3, $f^{-1}(\lambda_i)'s$ are fuzzy second category sets in (X,T). Also $\vee_{i=1}^{\infty}(\lambda_i) = 1$, in (Y,S) implies that $f^{-1}[\vee_{i=1}^{\infty}(\lambda_i)] = f^{-1}(1) = 1$. Now $f^{-1}[\vee_{i=1}^{\infty}(\lambda_i)] = \vee_{i=1}^{\infty}[f^{-1}(\lambda_i)]$, implies that $\vee_{i=1}^{\infty}[f^{-1}(\lambda_i)] = 1$ in (X,T). Thus, $\vee_{i=1}^{\infty}[f^{-1}(\lambda_i)] = 1$, where $[f^{-1}(\lambda_i)]'s$ are fuzzy second category sets in (X,T).

Proposition 4.5: If $f:(X,T) \to (Y,S)$ is a somewhat fuzzy continuous function from a fuzzy Baire space (X,T) onto a fuzzy compact space (Y,S), then $\vee_{j=1}^{N}(\mu_j) = 1$ where $(\mu_j)'s$ are fuzzy second category sets in (X,T).

Proof: Let the fuzzy topological space (Y,S) be a fuzzy compact space. Then, if $\vee_{i\in I}(\lambda_i) = 1$, where $\lambda_i \in S$, then there are finitely many indices $i_1, i_2, ...i_N \in I$ such that $\vee_{j=1}^{N}(\lambda_{i_j}) = 1$. Since f is a somewhat fuzzy continuous function from the fuzzy Baire space (X,T) onto (Y,S) and $(\lambda_{i_j})'s$ $(j = 1 to N)$ are fuzzy open sets in (Y,S), by proposition 4.3, $[f^{-1}(\lambda_{i_j})]'s$ are fuzzy second category sets in (X,T). Now $\vee_{j=1}^{N}(\lambda_{i_j}) = 1$ implies that $f^{-1}[\vee_{j=1}^{N}(\lambda_{i_j})] = f^{-1}(1) = 1$ and hence $\vee_{j=1}^{N}[f^{-1}(\lambda_{i_j})] = 1$. By putting $f^{-1}(\lambda_{i_j}) = \mu_j$ we have $\vee_{j=1}^{N}(\mu_j) = 1$, where $(\mu_j)'s$ are fuzzy second category sets in (X,T).

Proposition 4.6: If $f:(X,T) \to (Y,S)$ is a somewhat fuzzy open function from a fuzzy topological space (X,T) into a fuzzy Baire space (Y,S) and μ is a fuzzy open set in (X,T) then $f(\mu)$ is a fuzzy second category set in (Y,S).

Proof : Let μ be a fuzzy open set in (X,T). Since f is a somewhat fuzzy open function and $\mu \in T$ and $\mu \neq 0$ implies that $int[f(\mu)] \neq 0$ in (Y,S). We claim that $f(\mu)$ is a fuzzy second category set in (Y,S). Assume the contrary. Suppose that $f(\mu)$ is a fuzzy first category set in (Y,S). Since (Y,S) is a fuzzy Baire Space, by theorem 2.1, $int[f(\mu)] = 0$ in (Y,S), a contradiction. Hence $f(\mu)$ must be a fuzzy second category set in (Y,S).

Proposition 4.7: Let (Y,S) be a fuzzy Baire space and $f:(X,T) \to (Y,S)$ is

(i) a fuzzy continuous function from a fuzzy topological space (X,T) onto (Y,S),

(ii) $f(\mu)$ is a fuzzy somewhere dense set in (Y,S) for each non-zero fuzzy open set μ in (X,T) and

(iii) $f^{-1}[cl(\lambda)] \leq cl[f^{-1}(\lambda)]$ in (X,T) for any fuzzy set λ in Y,

then (X,T) is a fuzzy Baire space.

Proof: Let δ be a fuzzy residual set in (Y,S). Since f is a fuzzy continuous function from (X,T) onto (Y,S) such that $f(\mu)$ is a fuzzy somewhere dense set in (Y,S) for each non-zero fuzzy open set μ in (X,T), by proposition 4.1, $f^{-1}(\delta)$ is a fuzzy residual set in (X,T). Then $f^{-1}(\delta) = \wedge_{i=1}^{\infty}(\delta_i)$, where $cl[int(\delta_i)] = 1$ in (X,T).

By hypothesis, for the fuzzy set δ in (Y,S), we have $f^{-1}[cl(\delta)] \leq cl[f^{-1}(\delta)]$——(1) in (X,T). Since (Y,S) is a fuzzy Baire space , by theorem 2.1, for the fuzzy residual set δ in (Y,S), $cl(\delta) = 1$ in (Y,S). Then from (1), $f^{-1}(1) \leq cl[f^{-1}(\delta)]$ in (X,T) and hence $1 \leq cl[f^{-1}(\delta)]$ in (X,T). That is., $cl[f^{-1}(\delta)] = 1$ in (X,T).

Now $1 - cl[f^{-1}(\delta)] = 0$ and hence $int[1 - f^{-1}(\delta)] = 0$ and thus $int[1 - \wedge_{i=1}^{\infty}(\delta_i)] = 0$ in (X,T). Then, $int[\vee_{i=1}^{\infty}(1 - \delta_i)] = 0$ where $int[cl(1-\delta_i)] = 1 - cl[int(\delta_i)] = 1 - 1 = 0$. Let $\mu_i = 1 - \delta_i$. Therefore, $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$ where $(\mu_i)'s$ are fuzzy nowhere dense sets in (X,T), implies that (X,T) is a fuzzy Baire space.

Proposition 4.8: Let (X,T) be a fuzzy Baire space and $f:(X,T) \to (Y,S)$ is

(i) a fuzzy open and one - one function from (X,T) onto a fuzzy topological space (Y,S),

(ii) $f^{-1}(\eta)$ is a fuzzy somewhere dense set in (X,T) for each non-zero fuzzy open set η in (Y,S),

(iii) $f[cl(\lambda)] \leq cl[f(\lambda)]$ in (Y, S) for any fuzzy set λ in X,

then (Y, S) is a fuzzy Baire space.

Proof: Let δ be a fuzzy residual set in (X, T). Since f is a fuzzy open and fuzzy one-one function from (X, T) onto (Y, S) such that $f^{-1}(\eta)$ is a fuzzy somewhere dense set in (X, T), for each non-zero fuzzy open set η in (Y, S), by proposition 4.2, $f(\delta)$ is a fuzzy residual set in (Y, S). Then $f(\delta) = \wedge_{i=1}^{\infty}(\delta_i)$, where $cl[int(\delta_i)] = 1$ in (Y, S).
By hypothesis, for the fuzzy set δ in (X, T) we have $f[cl(\delta)] \leq cl[f(\delta)]$———(1). Since (X, T) is a fuzzy Baire space, by theorem 2.1, for the fuzzy residual set δ in (X, T), $cl(\delta) = 1$ in (X, T). Then from (1), $f(1) \leq cl[f(\delta)]$ in (Y, S) and hence $1 \leq cl[f(\delta)]$ in (Y, S). That is., $cl[f(\delta)] = 1$ in (Y, S). Then $1 - cl[f(\delta)] = 0$ and hence $int[1 - f(\delta)] = 0$ in (Y, S). Thus $int[1 - \wedge_{i=1}^{\infty}(\delta_i)] = 0$ in (Y, S) and hence $int[\vee_{i=1}^{\infty}(1 - \delta_i)] = 0$, where $int[cl(1 - \delta_i)] = 1 - cl[int(\delta_i)] = 1 - 1 = 0$. Let $\mu_i = 1 - \delta_i$. Therefore, $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$ where $int[cl(\mu_i)] = 0$, implies that (Y, S) is a fuzzy Baire space.

5. Conclusion:

In this paper the preservation of the fuzzy residuality of fuzzy sets under fuzzy continuous and fuzzy open functions between fuzzy topological spaces are established. A condition for a fuzzy Baire space to become a fuzzy resolvable space is also obtained.

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