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On fixed point theorems in dislocated quasi b-metric spaces

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ABSTRACT. The purpose of this paper is to introduce the common fixed point theorems in dislocated quasi b-metric spaces. These generalize the existing fixed point results in such spaces.

1. Introduction

Fixed point theory has been studied extensively from last few decades. Banach [7] established the well-known fixed point theorem for contraction mappings in metric space. Later, the several generalizations of this theorem in metric spaces have been appeared in the literature. Many authors invented the generalizations of metric spaces by various ways and established the contraction mapping theorem in it. A few examples of generalized metric spaces are quasi metric spaces, cone metric spaces,G-metric spaces, dislocated metric spaces,b-metric spaces,dislocated quasi metric spaces etc. On the similar lines, Banach contraction principle also extended by defining the new types of contraction principles in metric spaces as well as in generalized structures. Some of them are Kannan contraction, Ciric contraction T-Kannan contraction, T-Banach contraction, cyclic contraction, Chetterjea type contraction, α-ϕ-contractive mappings etc.

Recently, Chakkrid and Cholatis[15] introduced concept of dislocated quasi b-metric space. In this new structure they proved some fixed point theorems for cyclic contractions. Also they have discussed the topological structure of dislocated quasi b-metric spaces and studied their properties. Mujeeb Ur Rahman and Muhammad Sarwar[21] coined the concept of dislocated quasi b-metric spaces and established fixed point theorems

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for Kannan and Chetterjea type contractions. Also we study recent development in dislocated metric spaces in [25, 16, 27, 20, 28, 24].

In this paper, we establish some fixed point theorems for different types of contractions in dislocated quasi b-metric spaces. These extends some well known fixed point theorems existing in the literature. We develop some common fixed point theorems in this spaces. Also, we prove the fixed points of α-admissible mappings in dislocated quasi b-metric spaces.

The notion of dislocated quasi metric space was initiated by F. M. Zeyada, G. H. Hassan and M. A. Ahmed [29]. It is a generalization due to P. Hitzler and A. K. Seda[9, 10] in dislocated metric.

Definition 1.1.[29] *Let X be a non-empty set. Consider mapping $d : X \times X \to [0, \infty)$ and following conditions:*

(i) $d(x, x) = 0, \forall x \in X.$

(ii) $d(x, y) = 0 = d(y, x) \Rightarrow x = y, \forall x, y \in X.$

(iii) $d(x, y) = d(y, x), \forall x, y \in X.$

(iv) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X.$

If d satisfies all conditions (i) to (iv) then d is called metric on X. If d satisfies only (ii),(iii) and (iv) then d is called dislocated metric on X. If d satisfies only (i),(ii) and (iv) then d is called quasi-metric on X. The concept of b-metric spaces was introduced by Bakhtin[6].

Definition 1.2.[6] *Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $k \geq 1$ be a constant such that:*

(i) $d(x, y) = 0 = d(y, x) \iff x = y, \forall x, y \in X.$

(ii) $d(x, y) = d(y, x), \forall x, y \in X.$

(iii) $d(x, y) \leq k[d(x, z) + d(z, y)], \forall x, y, z \in X.$

Then pair (X, d) is called $b-$metric space.

After that Shah and Huassain[11] coined the concept of quasi-b-metric spaces to generalize b-metric spaces and obtained some fixed point theorems in such spaces.

Definition 1.3.[11] *Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $k \geq 1$ be a constant such that:*

(i) $d(x, y) = 0 = d(y, x) \iff x = y, \forall x, y \in X.$

(ii) $d(x, y) \leq k[d(x, z) + d(z, y)], \forall x, y, z \in X.$

Then pair (X, d) is called quasi b-metric space.

Alghamdi,Husasain and Salimi[5] introduced concept of b-metric-like spaces or dislocated b-metric spaces to generalize metric-like spaces.

Definition 1.4.[5] *Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $k \geq 1$ be a constant such that:*

(i) $d(x, y) = 0 \Rightarrow x = y, \forall x, y \in X.$

(ii) $d(x, y) = d(y, x), \forall x, y \in X.$

(iii) $d(x, y) \leq k[d(x, z) + d(z, y)], \forall x, y, z \in X.$

Then pair (X, d) is called a b-metric-like space.

Chakkrid and Cholatis[15] introduced concept of dislocated quasi-$b-$metric space as follows

Definition 1.5.[15] *Let X be a non-empty set. Let the mapping $d : X \times X \to [0, \infty)$ and constant $k \geq 1$ satisfy following conditions:*

(i) $d(x, y) = 0 = d(y, x) \Rightarrow x = y, \forall x, y \in X.$

(ii) $d(x, y) \leq k[d(x, z) + d(z, y)], \forall x, y, z \in X.$

Then the pair (X, d) is called dislocated quasi b-metric space or in short $dqb-$metric space.

The constant k is called coefficient of (X, d). It is clear that $b-$ metric spaces, quasi-$b-$metric spaces and $b-$metric-like spaces are $dqb-$metric spaces but converse is not true.

Example 1.6. *Let $X = R^+$ and for $p > 1, d : X \times X \to [0, \infty)$ be defined as,*

$$d(x, y) = |x - y|^p + |x|^p, \forall x, y \in X.$$

Then (X, d) is $dqb-$metric space with $k = 2^p > 1$. But (X, d) is not $b-$metric space and also not dislocated quasi metric space.

Example 1.7. *Let $X = R$ and suppose,*

$$d(x, y) = |2x - y|^2 + |2x + y|^2$$

then (X, d) is $dqb-$metric space with coefficient $k = 2$ but (X, d) is not a quasi-$b-$metric space. Also (X, d) is not dislocated quasi metric space.

Definition 1.8.[15] *Let (X, d) be a dqb-metric space. A sequence $\{x_n\}$ in a X is called to be dqb-converges to $x \in X$ if*

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

In this case x is called $dqb-$limit of $\{x_n\}$ and is written as $x_n \to x$.

Definition 1.9.[15] *Let (X, d) be a dqb-metric space. A sequence $\{x_n\}$ in a X is called as dqb-Cauchy sequence if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{n,m\to\infty} d(x_m, x_n).$$

Definition 1.10.[15] *A dqb-metric space (X, d) is said to be dqb-complete if every dqb-Cauchy sequence in it is dqb-convergent in X.*

Proposition 1.11.[15] *Every subsequence of a dqb-convergent sequence in a $dqb-$metric space (X, d) is dqb-convergent sequence.*

Proposition 1.12.[15] *Every subsequence of a dqb-Cauchy sequence in a dqb-metric space (X, d) is dqb-Cauchy sequence.*

Proposition 1.13.[15] *If (X, d) is a dqb-metric space then a function $f : X \to X$ is continuous if and only if $x_n \to x \Rightarrow fx_n \to fx$.*

Lemma 1.14.[15] *Limit of a dqb-convergent sequence in dqb-metric space is unique.*

Lemma 1.15. *Let be a dqb-metric space and $\{x_n\}$ be a sequence in it such that,*

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n), n = 1, 2, 3, \cdots$$

and $0 \leq \alpha k < 1, \alpha \in [0,1)$ where k is coefficient of (X,d), then $\{x_n\}$ is a Cauchy sequence in X.

Proposition 1.16.[15] *If u is limit of some dqb-convergent sequence in a dqb-metric space (X,d) then $d(u,u) = 0$.*

Proof. $0 = \lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(x_n, x_m) = d(u,u), m > n.$

Proposition 1.17.[15] *Every dqb-convergent sequence in a dqb-metric space (X,d) is dqb-Cauchy sequence.*

Definition 1.18.[14] *Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.*

Definition 1.19.[14] *Let f and g be self maps of a set X. Then f and g are said to be weakly compatible if they commute at their coincidence point.*

Proposition 1.20.[4] *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 1.21.[15] *Let $f : X \to X$ be a self mapping of X, f is said to be dqb$-$subsequentially convergent if for every sequence $\{x_n\}$, if fx_n is dqb$-$convergent then $\{x_n\}$ has a dqb$-$convergent subsequence in X.*

Definition 1.22.[15] *Let $f : X \to X$ be a self mapping of X, f is said to be dqb$-$sequentially convergent if for every sequence $\{x_n\}$, if fx_n is dqb$-$convergent then $\{x_n\}$ is also dqb$-$convergent in X.*

Definition 1.23.[22] *Let T be a self on a set X and $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is α-admissible mapping, if*

$$x, y \in X \text{ then } \ \alpha(x,y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.24.[19] *Let (X,d) be a dqb-metric space and $T, S : X \to X$ be two mappings,then the mapping S is called T-Banach contraction if there exists $\alpha \in [0,1)$ such that,*

$$d(TSx, TSy) \leq \alpha d(Tx, Ty), \forall x, y \in X.$$

Definition 1.25.[19] *Let (X,d) be a dqb-metric space and $T, S : X \to X$ be two mappings, then the mapping S is called T-Kannan contraction if there exists $\alpha \in [0,1/2)$ such that,*

$$d(TSx, TSy) \leq \alpha[d(Tx, TSx) + d(Ty, TSy)], \forall x, y \in X.$$

2. Main Results

We establish the following results

Theorem 2.1. *Let (X,d) be a dqb-complete metric space and let $f, g : X \to X$ be self-mappings satisfying the inequality*

$$d(fx, fy) \leq \alpha d(gx, gy), \forall x, y \in X \tag{2.1}$$

where $\alpha \in [0,1)$ such that $\alpha k \leq 1$ and k is coefficient of (X,d).

If $f(X) \subseteq g(X)$ and $g(X)$ is dqb-complete subspace of X, then f and g have unique point of coincidence in X. In addition if f and g are weakly compatible, then f and g have common unique fixed point in X.

Proof. Let x_0 be any arbitrary point in X. As $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Again we can choose $x_2 \in X$ such that $fx_1 = gx_2$. Continuing this procedure, we get a sequence $x_n \in X$ such that $fx_n = gx_{n+1}$, for $n = 0, 1, 2, 3, \cdots$.

Now consider,

$$d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) \leq \alpha d(gx_n, gx_{n-1}). \tag{2.2}$$

Making use of (2.2) repeatedly, we get,

$$d(gx_{n+1}, gx_n) \leq \alpha^n d(gx_1, gx_0). \tag{2.3}$$

Similarly we can prove,

$$d(gx_n, gx_{n+1}) \leq \alpha^n d(gx_0, gx_1).$$

Now, for $m, n \in N$ and $n > m$, consider,

$$\begin{aligned} d(gx_n, gx_m) &\leq k^{n-m} d(gx_n, gx_{n-1}) + k^{n-m-1} d(gx_{n-1}, gx_{n-2}) + \cdots + kd(gx_{m+1}, gx_m) \\ &\leq [k^{n-m}\alpha^{n-1} + k^{n-m-1}\alpha^{n-2} + \cdots + k\alpha^m] d(gx_1, gx_0) \\ &= [(k\alpha)^{n-m}\alpha^{m-1} + (k\alpha)^{n-m-1}\alpha^{m-1} + \cdots + (k\alpha)\alpha^{m-1}] d(gx_1, gx_0) \\ &\leq (n-m)\alpha^{m-1} d(gx_1, gx_0) \\ &< \alpha^{m-1}\eta, \end{aligned}$$

where $\eta > (n-m)d(gx_1, gx_0)$. Letting $m, n \to \infty$, we get,

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = 0.$$

Similarly for $m, n \in N$ and $n > m$, we can prove, $\lim_{m,n\to\infty} d(gx_m, gx_n) = 0$. Thus

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = \lim_{m,n\to\infty} d(gx_m, gx_n) = 0.$$

Hence $\{gx_n\}$ is a dqb-Cauchy sequence in X. Since $g(X)$ is dqb-complete, there exists $v \in g(X)$ such that $gx_n \to v$ as $n \to \infty$. Since $v \in g(X)$, we can find $u \in X$ such that $gu = v$.

Now,

$$d(gx_n, fu) = d(fx_{n-1}, fu) \leq \alpha d(gx_{n-1}, gu).$$

Letting $n \to \infty$ in above inequality,

$$\lim_{n\to\infty} d(gx_n, fu) \leq \alpha \lim_{n\to\infty} d(gx_{n-1}, gu) = 0.$$

This implies that,

$$\lim_{n\to\infty} d(gx_n, fu) = 0.$$

Similarly we can prove that,

$$\lim_{n\to\infty} d(fu, gx_n) = 0.$$

Hence, we conclude that $gx_n \to fu$ as $n \to \infty$. By uniqueness of limit in $dqb-$metric space, we get $fu = gu$. Thus $fu = gu$ is point of coincidence of f and g in X. Now we claim that the point of coincidence of f and g in X is unique. On the contrary, we assume that, there exists $\omega \in X$ such that $f\omega = g\omega$. Now,

$$d(gu, g\omega) = d(fu, f\omega) \leq \alpha d(gu, g\omega).$$

which gives contradiction unless, $d(gu, g\omega) = 0$. Similarly, we can get, $d(g\omega, gu) = 0$. Thus, $d(gu, g\omega) = d(g\omega, gu) = 0 \Rightarrow gu = g\omega$ and point of coincidence of f and g in X is unique. Using Proposition 1.20, the mappings f and g have unique common fixed point in X. □

Theorem 2.2. *Let (X, d) be a dqb-complete metric space and let $f, g : X \to X$ be self-mappings satisfying the inequality*

$$d(fx, fy) \leq \alpha[d(fx, gx) + d(fy, gy)], \forall x, y \in X, \tag{2.4}$$

where $\alpha \in [0, 1/2)$ such that $k\frac{\alpha}{1-\alpha} < 1$ and k is coefficient of (X, d).

If $f(X) \subseteq g(X)$ and $g(X)$ is dqb-complete subspace of X, then f and g have unique point of coincidence in X. In addition if f and g are weakly compatible, then f and g have common unique fixed point in X.

Proof. Let x_0 be any arbitrary point in X. As $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Again we can choose $x_2 \in X$ such that $fx_1 = gx_2$. Repeating in the same manner for $x_n \in X$ we can choose $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$, for $n = 0, 1, 2, 3, \cdots$. Now consider,

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\ &\leq \alpha[d(fx_n, gx_n) + d(fx_{n-1}, gx_{n-1}) \\ &= \alpha[d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]. \end{aligned}$$

This implies that

$$d(gx_{n+1}, gx_n) \leq \frac{\alpha}{1-\alpha} d(gx_n, gx_{n-1}).$$

Since $h = \frac{\alpha}{1-\alpha} < 1$, then,

$$d(gx_{n+1}, gx_n) \leq hd(gx_n, gx_{n-1}).$$

Continuing this way, we get for $n \in N$,

$$d(gx_{n+1}, gx_n) \leq h^n d(gx_1, gx_0).$$

Now, for $m, n \in N$ and $n > m$, we have,

$$\begin{aligned} d(gx_n, gx_m) &\leq k^{n-m} d(gx_n, gx_{n-1}) + k^{n-m-1} d(gx_{n-1}, gx_{n-2}) + \cdots + kd(gx_{m+1}, gx_m) \\ &\leq [k^{n-m}h^{n-1} + k^{n-m-1}h^{n-2} + \cdots + kh^m] d(gx_1, gx_0) \\ &= [(kh)^{n-m}h^{m-1} + (kh)^{n-m-1}h^{m-1} + \cdots + (kh)h^{m-1}] d(gx_1, gx_0) \\ &\leq (n-m)h^{m-1} d(gx_1, gx_0) \\ &< h^{m-1}\eta, \end{aligned}$$

where $\eta > (n-m)d(gx_1, gx_0)$. Letting limit as $m, n \to \infty$, we get,

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = 0.$$

Similarly for $m, n \in N$ and $n > m$, we can prove that,

$$\lim_{m,n\to\infty} d(gx_m, gx_n) = 0.$$

Thus

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = \lim_{m,n\to\infty} d(gx_m, gx_n) = 0.$$

Hence $\{gx_n\}$ is a dqb-Cauchy sequence in X. Since $g(X)$ is dqb-complete, there exists $v \in g(X)$ such that $gx_n \to v$ as $n \to \infty$. Since $v \in g(X)$, we can find $u \in X$ such that $gu = v$. Consider,

$$\begin{aligned} d(gx_n, fu) &= d(fx_{n-1}, fu) \\ &\le \alpha[d(fx_{n-1}, gx_{n-1}) + d(fu, gu)] \\ &= \alpha[d(gx_n, gx_{n-1}) + d(fu, gu)]. \end{aligned}$$

Letting limit as $n \to \infty$, we have

$$d(gu, fu) = d(v, fu) \le \alpha d(fu, gu). \tag{2.5}$$

Similarly we can prove that,

$$d(fu, gu) \le \alpha d(gu, fu). \tag{2.6}$$

From (2.5)and (2.6), we have

$$d(gu, fu) \le \alpha d(fu, gu) \le \alpha^2 d(gu, fu).$$

Since $\alpha \in [0, 1/2]$, above inequality is true only if

$$d(gu, fu) = d(fu, gu) = 0.$$

Hence we conclude that $fu = gu$. This shows that u is point of coincidence of f and g in X. Now we claim that this point of coincidence of f and g in X is unique. On the contrary, we assume that there exists $\omega \in X$ such that $f\omega = g\omega$.

Now,

$$d(gu, g\omega) = d(fu, f\omega) \le \alpha[d(fu, gu) + d(f\omega, g\omega).$$

Using Proposition 1.16 we get $d(gu, g\omega) = 0$. Similarly we get, $d(g\omega, gu) = 0$. Thus, $d(gu, g\omega) = d(g\omega, gu) = 0 \Rightarrow gu = g\omega$ and point of coincidence of f and g in X is unique. Hence by Proposition 1.20, f and g have unique common fixed point in X. □

Theorem 2.3. *Let (X, d) be a dqb-complete metric space with coefficient $k \ge 1$. Let $f, T : X \to X$ be self-mappings such that T is continuous,one-one and f is continuous T-Banach contraction with $k\alpha \le 1$. If T is dqb sub-sequentially convergent then f has unique fixed point in X.*

Proof. Let x_0 be any arbitrary point in X. Define $\{x_n\}$ by,

$$x_1 = fx_0, x_2 = fx_1, \cdots, x_{n+1} = fx_n = f^n x_0, \cdots,\ n = 0, 1, 2, 3, \cdots.$$

Since f is T-Banach contraction with $k\alpha \leq 1$, we have,

$$\begin{aligned} d(Tf^{n+1}x_0, Tf^n x_0) &\leq \alpha d(Tf^n x_0, Tf^{n-1}x_0) \\ &\leq \alpha^2 d(Tf^{n-1}x_0, Tf^{n-2}x_0). \end{aligned}$$

Continuing in this way,we get

$$d(Tf^{n+1}x_0, Tf^n x_0) \leq \alpha^n d(Tfx_0, Tx_0).$$

Similarly, we can show

$$d(Tf^n x_0, Tf^{n+1}x_0) \leq \alpha^n d(Tx_0, Tfx_0).$$

Let $\eta = \max\{d(Tx_0, Tfx_0), d(Tfx_0, Tx_0)\}$. Thus,

$$d(Tf^{n+1}x_0, Tf^n x_0) \leq \alpha^n \eta,$$

and

$$d(Tf^n x_0, Tf^{n+1}x_0) \leq \alpha^n \eta.$$

For $m, n \in N$ such that $n > m$, we have,

$$\begin{aligned} d(Tf^n x_0, Tf^m x_0) &\leq [k^{n-m} d(Tf^n x_0, Tf^{n-1}x_0) + k^{n-m-1} d(Tf^{n-1}x_0, Tf^{n-2}x_0) + \cdots + kd(Tf^{m+1}x_0, Tf^m x_0)] \\ &\leq [k^{n-m}\alpha^{n-1} + k^{n-m-1}\alpha^{n-2} + \cdots + k\alpha^m]\eta \\ &= [(k\alpha)^{n-m}\alpha^{m-1} + (k\alpha)^{n-m-1}\alpha^{m-1} + \cdots + (k\alpha)\alpha^{m-1}]\eta \\ &\leq \alpha^{m-1}(n-m)\eta \\ &< \alpha^{m-1}\xi. \end{aligned}$$

where $\xi > (n-m)\eta$. Letting limit as $m, n \to \infty$,

$$\lim_{m,n\to\infty} d(Tf^n x_0, Tf^m x_0) = 0.$$

Similarly for $m, n \in N$ such that $n > m$, we can prove

$$\lim_{m,n\to\infty} d(Tf^m x_0, Tf^n x_0) = 0.$$

Thus

$$\lim_{m,n\to\infty} d(Tf^n x_0, Tf^m x_0) = 0 = \lim_{m,n\to\infty} d(Tf^m x_0, Tf^n x_0).$$

Therefore $\{Tf^n x_0\}$ is dqb-Cauchy sequence in X. Since (X, d) is dqb-complete there exists $v \in X$ such that,

$$\lim_{n\to\infty} d(Tf^n x_0, v) = 0 = \lim_{n\to\infty} d(v, Tf^n x_0).$$

Since T is dqb-sub sequentially convergent,$\{f^n x_0\}$ has a dqb-convergent subsequence $\{f^{n_i}x_0\}$ such that

$$f^{n_i}x_0 \to u \text{ as } i \to \infty.$$

Since T and f are continuous, then $Tf^{n_i}x_0 \to Tu$ as $i \to \infty$ and $f^{n_i+1}x_0 \to fu$ as $i \to \infty$. This implies that

$$Tf^{n_i+1}x_0 \to Tfu \text{ as } i \to \infty.$$

By uniqueness of limit in dqb-metric space (X, d), we get $Tu = Tfu$. Since T is one-one, we have $fu = u$. Thus $u \in X$ is fixed point of f. To will prove this fixed point of f is unique. We assume that $\omega \in X$ is another fixed point of f i.e. $f\omega = \omega$. Consider,

$$d(Tfu, Tf\omega) \leq \alpha d(Tu, T\omega)$$

That is

$$d(Tu, T\omega) \leq \alpha d(Tu, T\omega).$$

Which is contradiction, since $\alpha \in [0, 1)$. Therefore $d(Tu, T\omega) = 0$. Similarly we can prove $d(T\omega, Tu) = 0$. Thus,

$$d(Tu, T\omega) = 0 = d(T\omega, Tu)$$

i.e. $Tu = T\omega$. But since T is one-one, so $u = \omega$. □

Theorem 2.4. *Let (X, d) be a dqb-complete metric space with coefficient $k \geq 1$. Let $f, T : X \to X$ be self-mappings such that T is continuous,one-one and f is continuous T-Kannan contraction with $k\alpha \leq 1$. If T is dqb-sub-sequentially convergent, then f has unique fixed point in X.*

Proof. Let x_0 be any arbitrary point in X. Define $\{x_n\}$ by,

$$x_1 = fx_0, x_2 = fx_1, \cdots, x_{n+1} = fx_n = f^n x_0, \cdots, n = 0, 1, 2, 3, \cdots.$$

Since f is T-Kannan contraction with $k\alpha \leq 1$, we have,

$$d(Tfx_0, Tf^2x_0) \leq \alpha[d(Tx_0, Tfx_0) + d(Tfx_0, Tf^2x_0)].$$

This gives

$$d(Tfx_0, Tf^2x_0) \leq \frac{\alpha}{1-\alpha} d(Tx_0, Tfx_0).$$

Let $h = \frac{\alpha}{1-\alpha} < 1$. We have

$$d(Tfx_0, Tf^2x_0) \leq hd(Tx_0, Tfx_0).$$

Now

$$d(Tf^2x_0, Tf^3x_0) \leq \alpha[d(Tfx_0, Tf^2x_0) + d(Tf^2x_0, Tf^3x_0)].$$

This implies that

$$\begin{aligned} d(Tf^2x_0, Tf^3x_0) &\leq \frac{\alpha}{1-\alpha} d(Tfx_0, Tf^2x_0) \\ &\leq hd(Tfx_0, Tf^2x_0) \\ &\leq h^2 d(Tx_0, Tfx_0). \end{aligned}$$

In general for any $n \in N$, we get,

$$d(Tf^n x_0, Tf^{n+1}x_0) \leq h^n d(Tx_0, Tfx_0).$$

Similarly we can prove,

$$d(Tf^{n+1}x_0, Tf^n x_0) \leq h^n d(Tfx_0, Tx_0).$$

Let $\eta = \max\{d(Tx_0, Tfx_0), d(Tfx_0, Tx_0)\}$. Now for $m, n \in N$ such that $n > m$, we have,

$$\begin{aligned}
d(Tf^n x_0, Tf^m x_0) &\leq [k^{n-m} d(Tf^n x_0, Tf^{n-1} x_0) + k^{n-m-1} d(Tf^{n-1} x_0, Tf^{n-2} x_0) + \cdots + kd(Tf^{m+1} x_0, Tf^m x_0)] \\
&\leq [k^{n-m} h^{n-1} + k^{n-m-1} h^{n-2} + \cdots + kh^m]\eta \\
&= [(kh)^{n-m} h^{m-1} + (kh)^{n-m-1} h^{m-1} + \cdots + (kh) h^{m-1}]\eta \\
&\leq h^{m-1}(n-m)\eta \\
&< h^{m-1}\xi.
\end{aligned}$$

where $\xi > (n-m)\eta$. Letting limit as $m, n \to \infty$,we have

$$\lim_{m,n\to\infty} d(Tf^n x_0, Tf^m x_0) = 0.$$

Similarly for $m, n \in N$ such that $n > m$, we can prove

$$\lim_{m,n\to\infty} d(Tf^m x_0, Tf^n x_0) = 0.$$

Thus

$$\lim_{m,n\to\infty} d(Tf^n x_0, Tf^m x_0) = 0 = \lim_{m,n\to\infty} d(Tf^m x_0, Tf^n x_0).$$

Therefore $\{Tf^n x_0\}$ is dqb-Cauchy sequence in X. Since (X, d) is dqb-complete, so there exists $v \in X$ such that,

$$\lim_{n\to\infty} d(Tf^n x_0, v) = 0 = \lim_{n\to\infty} d(v, Tf^n x_0).$$

Since T is $dqb-$subsequentially convergent, $\{f^n x_0\}$ has a dqb-convergent subsequence $\{f^{n_i} x_0\}$ such that

$$f^{n_i} x_0 \to u \text{ as } i \to \infty.$$

Since T and f are continuous, then

$$Tf^{n_i} x_0 \to Tu \text{ as } i \to \infty$$

and

$$f^{n_i+1} x_0 \to fu \text{ as } i \to \infty.$$

This shows

$$Tf^{n_i+1} x_0 \to Tfu \text{ as } i \to \infty.$$

By uniqueness of limit in dqb-metric space (X, d), we get $Tu = Tfu$. Since T is one-one, so $fu = u$. Thus $u \in X$ is fixed point of f. We will prove this fixed point of f is unique. Assume that $\omega \in X$ is another fixed point of f i.e. $f\omega = \omega$. Consider,

$$d(Tfu, Tf\omega) \leq \alpha[d(Tu, Tfu) + d(T\omega, Tf\omega).$$

That is

$$d(Tu, T\omega) \leq \alpha[d(Tu, Tu) + d(T\omega, T\omega)].$$

This implies $d(Tu, T\omega) = 0$. Similarly we can show $d(T\omega, Tu) = 0$. Thus, $d(Tu, T\omega) = 0 = d(T\omega, Tu)$. Therefore, $Tu = T\omega$. But since T is one-one, so $u = \omega$. $\square$

Theorem 2.5. *Let (X, d) be a dqb-complete metric space with coefficient $k \geq 1$. Let $f : X \to X$ be continuous self-mapping satisfying*

$$d(fx, fy) \leq \alpha\{d(x, y), d(fx, x), d(y, fy)\}, \forall x, y \in X,$$

where $\alpha \in [0, 1)$ such that $k\alpha \leq 1$. Then f has a unique fixed point in X.

Proof. Let x_0 be any arbitrary point in X. Define $\{x_n\}$ by,

$$x_1 = fx_0, x_2 = fx_1, \cdots, x_{n+1} = fx_n, \cdots, n = 0, 1, 2, 3, \cdots.$$

Consider,

$$\begin{aligned} d(x_1, x_2) = d(fx_0, fx_1) &\leq \alpha \max\{d(x_0, x_1), d(fx_0, x_0), d(x_1, fx_1)\} \\ &= \alpha \max\{d(x_0, x_1), d(x_1, x_0), d(x_1, x_2)\} \\ &\leq \alpha \max\{d(x_0, x_1), d(x_1, x_0)\}. \end{aligned}$$

Let $\max\{d(x_0, x_1), d(x_1, x_0)\} = \eta$. Then $d(x_1, x_2) \leq \alpha\eta$. Similarly we can show $d(x_2, x_1) \leq \alpha\eta$.

Now,

$$\begin{aligned} d(x_2, x_3) &\leq \alpha \max\{d(x_1, x_2), d(fx_1, x_1), d(x_2, fx_2)\} \\ &= \alpha \max\{d(x_1, x_2), d(x_2, x_1), d(x_2, x_3)\} \\ &\leq \alpha \max\{d(x_1, x_2), d(x_2, x_1)\} \\ &\leq \alpha^2\eta. \end{aligned}$$

Similarly, $d(x_3, x_2) \leq \alpha^2\eta$. In general, for any $n \in N$, we get,

$$d(x_n, x_{n+1}) \leq \alpha^n\eta,$$

and

$$d(x_{n+1}, x_n) \leq \alpha^n\eta.$$

Consider, for any $m, n \in N, n > m$,

$$\begin{aligned} d(x_n, x_m) &\leq k^{n-m}d(x_n, x_{n-1}) + k^{n-m-1}d(x_{n-1}, x_{n-2}) + ... + kd(x_{m+1}, x_m) \\ &\leq [k^{n-m}\alpha^{n-1} + k^{n-m-1}\alpha^{n-2} + ... + k\alpha^m]\eta \\ &= [(k\alpha)^{n-m}\alpha^{m-1} + (k\alpha)^{n-m-1}\alpha^{m-1} + ... + (k\alpha)\alpha^{m-1}]\eta \\ &\leq (n-m)\alpha^{m-1}\eta \\ &< \alpha^{m-1}\beta, \quad \beta > (n-m)\eta. \end{aligned}$$

Taking limit as $m, n \to \infty$, we get, $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. Similarly for any $m, n \in N, n > m$, we can prove that, $\lim_{m,n\to\infty} d(x_m, x_n) = 0$. Thus,

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0 = lim_{m,n\to\infty} d(x_m, x_n).$$

Therefore $\{x_n\}$ is a dqb-Cauchy sequence in X. Since (X,d) is dqb-complete, there exists $u \in X$ such that $x_n \to u$. But f is continuous therefore $fx_n \to u$ i.e. $x_{n+1} \to fu$. By uniqueness of limit in dqb-metric space (X,d), we get, $fu = u$. That is u is fixed point of f in X. To prove that this fixed point of f is unique, we assume that $\omega \in X$ is another fixed point of f i.e. $f\omega = \omega$.

Consider,

$$\begin{aligned} d(u,\omega) = d(fu, f\omega) &\leq \alpha \max d(u,\omega), d(fu,u), d(\omega, f\omega) \\ &= \alpha \max d(u,\omega), d(u,u), d(\omega,\omega) \end{aligned}$$

Using Proposition 1.16 we get, $d(u,\omega) = 0$. Similarly we can prove that $d(\omega,u) = 0$. Thus, $d(u,\omega) = 0 = d(\omega,u)$. Therefore,$u = \omega$. Thus uniqueness of fixed point of f is proved and hence the theorem. □

Theorem 2.6. *Let (X,d) be a dqb-complete metric space with coefficient $k \geq 1$ and let $f,g : X \to X$ be self-mappings satisfying the inequality*

$$\phi(d(fx,fy)) \leq \phi(d(gx,gy))^{\alpha}, \forall x,y \in X, \tag{2.7}$$

where $\alpha \in (0,1)$ such that $\alpha k \leq 1$ and $\phi : [0,\infty) \to [1,\infty)$ is non-decreasing function satisfying,

(i) $\phi(t) = 1 \iff t = 0$.

(ii) for every sequence $\{t_n\}$ *in*$(0,\infty), \phi(t_n) \to 1 \iff t_n \to 0$ *asn* $\to \infty$.

(iii) $\phi(t+s) \leq \phi(t)\phi(s) \forall t,s > 0$.

If $f(X) \subseteq g(X)$ and $g(X)$ is dqb-complete subspace of X, then f and g have unique point of coincidence in X. In addition if f and g are weakly compatible, then f and g have common unique fixed point in X.

Proof. Let x_0 be any arbitrary point in X. As $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Again we can choose $x_2 \in X$ such that $fx_1 = gx_2$. Repeating in the same manner for $x_n \in X$ we can choose $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$, for $n = 0,1,2,3,\cdots$. Consider,

$$\phi(d(gx_1, gx_2)) = \phi(d(fx_0, fx_1)) \leq \phi(d(gx_0, gx_1))^{\alpha}.$$

And

$$\phi(d(gx_2, gx_3)) = \phi(d(fx_,fx_2)) \leq \phi(d(gx_1, gx_2))^{\alpha} \leq \phi(d(gx_0, gx_1))^{\alpha^2}.$$

In general for any $n \in N$, we get,

$$\phi(d(gx_n, gx_{n+1})) \leq \phi(d(gx_0, gx_1))^{\alpha^n}.$$

Similarly, we can prove that,

$$\phi(d(gx_{n+1}, gx_n)) \leq \phi(d(gx_1, gx_0))^{\alpha^n}.$$

Take $\delta = \max\{d(gx_0, gx_1), d(gx_1, gx_0)\}$, then,

$$\begin{aligned} \phi(d(gx_n, gx_{n+1})) &\leq \phi(\delta)^{\alpha^n}, \\ \phi(d(gx_{n+1}, gx_n)) &\leq \phi(\delta)^{\alpha^n}. \end{aligned}$$

For $m, n \in N, n > m$, consider,

$$\begin{aligned}\phi(d(gx_n, gx_m)) &\leq \phi(k^{n-m}d(gx_n, gx_{n-1}))\phi(k^{n-m-1}d(gx_{n-1}, gx_{n-2}))\cdots\phi(kd(gx_{m+1}, gx_m))\\ &\leq \phi(\delta)^{\alpha^{n-1}}\phi(\delta)^{\alpha^{n-2}}\cdots\phi(\delta)^{\alpha^m}\\ &= \phi(\delta)^{\alpha^{n-1}+\alpha^{n-2}+\cdots+\alpha^m}\\ &\leq \phi(\delta)^{(n-m)\alpha^m}.\end{aligned}$$

Taking limit as $m, n \to \infty$, we get, $\lim_{m,n\to\infty} \phi(d(gx_n, gx_m)) = 1$, which implies that

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = 0.$$

Similarly, we can prove that for $m, n \in N, n > m$,

$$\lim_{m,n\to\infty} d(gx_m, gx_n) = 0.$$

Thus,

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = \lim_{m,n\to\infty} d(gx_m, gx_n) = 0.$$

Hence $\{gx_n\}$ is a Cauchy sequence in X. Since $g(X)$ is dqb-complete, there exists $v \in g(X)$ such that $gx_n \to v$ as $n \to \infty$. we can find $u \in X$ such that $gu = v$.

Now consider,

$$\phi(d(gx_n, fu)) = \phi(d(fx_{n-1}, fu)) \leq \phi(d(gx_{n-1}, gu))^{\alpha}.$$

Letting $n \to \infty$ in above inequality, we get

$$\lim_{n\to\infty} \phi(d(gx_n, fu)) \leq 1,$$

which implies that

$$\lim_{n\to\infty} d(gx_n, fu) = 0.$$

Similarly, we can prove that, $\lim_{n\to\infty} d(fu, gx_n) = 0$. Thus,

$$\lim_{n\to\infty} d(gx_n, fu) = 0 = lim_{n\to\infty}d(fu, gx_n).$$

Hence we conclude that $gx_n \to fu$ as $n \to \infty$. By uniqueness of limit in dqb-metric space, we get $fu = gu$.

Thus $fu = gu$ is point of coincidence of f and g in X. Now we prove that this point of coincidence of f and g in X is unique. On the contrary we assume that there exists $\omega \in X$ such that $f\omega = g\omega$. We assume that $d(gu, g\omega) \neq 0$.

Now,

$$\phi(d(gu, g\omega)) = \phi(d(fu, f\omega)) \leq \phi(d(gu, g\omega))^{\alpha} < \phi(d(gu, g\omega)),$$

which gives contradiction unless, $d(gu, g\omega) = 0$. Similarly, we get, $d(g\omega, gu) = 0$. Thus, $d(gu, g\omega) = d(g\omega, gu) = 0 \Rightarrow gu = g\omega$ and point of coincidence of f and g in X is unique. Hence by Proposition 1.20, f and g have unique common fixed point in X. □

Theorem 2.7. *Let (X,d) be a dqb-complete metric space with coefficient $k \geq 1$ and let $f : X \to X$ be $\alpha-$admissible, continuous self-mapping satisfying the inequality*

$$\alpha(x,y)d(fx,fy) \leq \phi(d(x,y)), \forall x,y \in X, \tag{2.8}$$

where $\phi : [0,\infty) \to [1,\infty)$ is non-decreasing function. If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq$, then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$. Define sequence $\{x_n\}$ in X by

$$x_1 = fx_0, x_2 = fx_1, \cdots, x_{n+1} = fx_n, \cdots, n = 0,1,2,3,\cdots.$$

Since f is $\alpha-$admissible mapping,

$$\alpha(x_0,x_1) = \alpha(x_0, fx_0) \geq 1 \Rightarrow \alpha(fx_0, fx_1) = \alpha(x_1,x_2) \geq 1.$$

Similarly we get, $\alpha(x_2,x_3) \geq 1$. In general for any $n \in N$, we get,

$$\alpha(x_n, x_{n+1}) \geq 1.$$

Now consider,

$$d(x_1,x_2) = d(fx_0, fx_1) \leq \alpha(x_0,x_1)d(fx_0,fx_1) \leq \phi(d(x_0,x_1)).$$

And $d(x_2,x_3) = d(fx_1,fx_2) \leq \alpha(x_2,x_3)d(fx_1,fx_2) \leq \phi(d(x_1,x_2)) \leq \phi^2(d(x_0,x_1))$. Repeating this way,for any $n \in N$, we get,

$$d(x_n,x_{n+1}) \leq \phi^n(d(x_0,x_1)).$$

In the same manner we can prove that,

$$d(x_{n+1},x_n) \leq \phi^n(d(x_1,x_0)).$$

Let $\delta = \{d(x_0,x_1), d(x_1,x_0)\}$, then,

$$d(x_n,x_{n+1}) \leq \phi^n(\delta),$$
$$d(x_{n+1},x_n) \leq \phi^n(\delta).$$

Now for fixed $\epsilon > 0$, we can find $K \in N$ such that $\sum_{n\geq K} \phi^n(d(x_0,x_1)) < \epsilon$. For $m,n \in N, n > m$,

$$\begin{aligned}
d(x_n,x_m) &\leq k^{n-m}d(x_n,x_{n-1}) + k^{n-m-1}d(x_{n-1},x_{n-2}) + ... + kd(x_{m+1},x_m) \\
&\leq [k^{n-m}\phi^{n-1}(\delta) + k^{n-m-1}\phi^{n-2}(\delta) + ... + k\phi^m(\delta)] \\
&\leq (n-m)k^{n-m} \sum_{i=m}^{n-m} \phi^i(\delta) \\
&\leq (n-m)k^{n-m} \sum_{n\geq K} \phi^n(\delta) \\
&< \epsilon(n-m)k^{n-m} \\
&< \eta.
\end{aligned}$$

Thus, $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. Similarly we can prove for $m, n \in N, n > m$, that, $\lim_{m,n\to\infty} d(x_m, x_n) = 0$. Thus,

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0 = \lim_{m,n\to\infty} d(x_m, x_n).$$

Hence $\{x_n\}$ is a dqb-Cauchy sequence in X. Since (X, d) is dqb-complete there exists $u \in X$ such that $x_n \to u$. Since f is continuous $fx_n \to fu$ i.e. $x_{n+1} \to fu$. By uniqueness of limit in dqb-metric space, we get $fu = u$, that is u is fixed point of f in X. □

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