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Ball convergence for Ostrowski-like method with accelerated eighth order convergence under weak conditions

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ABSTRACT. We present a local convergence analysis of an optimal eighth-order family of Ostrowski-like methods for approximating a locally unique solution of a nonlinear equation. Earlier studies have shown convergence of these methods under hypotheses up to the fifth derivative of the function although only the first derivative appears in the method. In this study, we expand the applicability of these methods using only hypotheses up to the first derivative of the function. This way the applicability of these methods is extended under weaker hypotheses. Moreover, the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented in this study.

1 Introduction

A lot of problems in computational sciences can be written using mathematical modelling [7, 9, 18, 25] in the form

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq S \to S$ is a differentiable nonlinear function and D is a convex subset of S ($S = \mathbb{R}$ or $\mathbb{C}$). The solution ξ of equation (1.1) can be found in explicit form only in special cases. That is why most solution methods for these equations are usually iterative. Newton-like methods are famous for finding the solution of

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(1.1). The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1-25].

In this paper, we study the local convergence of eighth-order family of methods [23] defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - A_n^{-1}F(y_n)F'(x_n)^{-1}F(x_n), \\ x_{n+1} = z_n - H(t_n)[x_n, z_n; F]^{-1}[x_n, y_n; F][y_n, z_n; F]^{-1}F(z_n), \end{cases} \tag{1.2}$$

where $x_0 \in D$ is an initial point, $H : S \to S$ is a continuous function, $t_n = \dfrac{F(z_n)}{F(x_n)}$, $A_n = F(x_n) - 2F(y_n)$, and $[\cdot, \cdot; F]$ denotes a divided difference of order one for F. The above method (1.2) utilizes four functional evaluations, viz. $F(x_n)$, $F'(x_n)$, $F(y_n)$ and $F(z_n)$ per step to achieve eighth order of convergence. Hence, efficiency index [25] of the established family is $E = \sqrt[4]{8} \approx 1.682$. The convergence of method (1.2) was shown using Taylor expansions and hypotheses reaching up to the fifth derivative of the function F although only first derivative appears in the method. We will show that method (1.2) is well-defined and convergent using hypotheses only on the first derivative. Notice that the method (1.2) was not shown to be well defined. However, the eighth order of convergence was shown assuming that method (1.2) is well defined which may not be the case. These hypotheses limit the applicability of method (1.2).

As a motivational example, define function F on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{1.3}$$

We have that

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x,$$
$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Clearly, function $F'''(x)$ is unbounded on D. Hence, the results in [23] cannot be applied to solve equation $F(x) = 0$, where F is given by (1.3). Moreover, the results in [23] do not provide computable convergence radii, error bounds on the distances $|x_n - x^*|$ and uniqueness of the solution results. We address all these problems using only hypotheses on the first derivative. We use the computational order of convergence (COC) to approximate the convergence order (which does not depend upon the solution ξ). Moreover, we present the results in a more general setting of a complex space.

The rest of the paper is organized as follows: In Section 2, we present the local convergence of method (1.2). The numerical examples are presented in the concluding Section 3.

2 Local convergence analysis

We present the local convergence analysis of method (1.2) in this section.
Let $L_0 > 0,\ L > 0$ and $M \geq 1$ be given parameters. Suppose that there exist a function $H : S \to S$ such that H is continuous nondecreasing function and

$$|H(t)| \leq |H(|t|)|. \tag{2.1}$$

The local convergence analysis that follows uses some scalar functions and parameters. Define functions $g_1,\ p$ and h_p on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} g_1(t) &= \frac{Lt}{2(1-L_0t)}, \\ p(t) &= \frac{L_0t}{2} + 2g_1(t), \\ h_p(t) &= p(t) - 1, \end{aligned}$$

and parameter r_1 by

$$r_1 = \frac{2}{2L_0 + L}.$$

We have that $g_1(r_1) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$. Then, we also get $h_p(0) = -1 < 0$ and $h_p(t) \to \infty$ as $t \to \frac{1^-}{L_0}$. It follows from the intermediate value theorem that function h_p has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_p the smallest such zero. Moreover, define functions $g_2, h_2, p_1, h_{p_1}, p_2, h_{p_2}$ on the interval $[0, r_p)$ by

$$\begin{aligned} g_2(t) &= \left(1 + \frac{M^2}{(1-L_0t)(1-p(t))}\right) g_1(t), \\ h_2(t) &= g_2(t) - 1, \\ p_1(t) &= \frac{L_0}{2}(1 + g_1(t))t, \\ h_{p_1}(t) &= p_1(t) - 1, \\ p_2(t) &= \frac{L_0}{2}(g_1(t) + g_2(t))t \end{aligned}$$

and $h_{p_2}(t) = p_2(t) - 1$.

Then, we get that $h_2(0) = h_{p_1}(0) = h_{p_2}(0) = -1 < 0$ and $h_2(t) \to +\infty, h_{p_1}(t) \to +\infty, h_{p_2}(t) \to +\infty$ as $t \to r_p^-$. Denote by r_2, r_{p_1}, r_{p_2} the smallest zeros of functions h_2, h_{p_1}, h_{p_2} on the interval $(0, r_p)$, respectively. Let $\bar{r} = min\{r_{p_1}, r_{p_2}\}$. Furthermore, define functions g_3 and h_3 on the interval $[0, \bar{r})$ by

$$g_3(t) = \left(1 + \frac{M^2(1+g_1(t))|H(\frac{Mg_2(t)}{1-\frac{L_0}{2}t})|t}{(1-p_1(t))(1-p_2(t))}\right) g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

We obtain that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \bar{r}^-$. Denote by r_3 the smallest zero of function h_3 in the interval $[0, \bar{r})$. Define the radius of convergence r by

$$r = min\{r_i\},\ i = 1, 2, 3. \tag{2.2}$$

Then, we have that

$$0 < r \leq r_1 < \frac{1}{L_0} \tag{2.3}$$

and for each $t \in [0, r)$

$$0 \leq g_i(t) < 1, \; i = 1, 2, 3 \tag{2.4}$$

$$0 \leq p_1(t) < 1 \tag{2.5}$$

and

$$0 \leq p_2(t) < 1. \tag{2.6}$$

Let $U(v, \rho)$ and $\bar{U}(v, \rho)$ stand respectively for the open and closed balls in S with center at $v \in S$ and of radius $\rho > 0$. Next, the local convergence analysis of method (1.2) shall be presented using previous notations.

Theorem 2.1 *Let $F : D \subseteq S \to S$ be a differentiable function. Suppose that there exists $\xi \in D$, $L_0 > 0$ such that for each $x \in D$*

$$F(\xi) = 0, \; F'(\xi) \neq 0 \tag{2.7}$$

and

$$|F'(\xi)^{-1}(F'(x) - F'(\xi))| \leq L_0|x - \xi|. \tag{2.8}$$

Moreover, suppose there exist $L > 0$ and $M \geq 1$ such that for each $x, y \in D_0 := D \cap U(\xi, \frac{1}{L_0})$

$$|F'(\xi)^{-1}(F'(x) - F'(y))| \leq L|x - y|, \tag{2.9}$$

$$|F'(\xi)^{-1}F'(x)| \leq M, \tag{2.10}$$

$$\bar{U}(\xi, r) \subseteq D, \tag{2.11}$$

and there exist a function $H : S \to S$ satisfying (2.1)*, where the convergence radius r is defined in* (2.2)*. Then, the sequence generated for $x_0 \in U(\xi, r) \backslash \{\xi\}$ by method* (1.2) *is well defined, remains in $U(\xi, r)$ for each $n = 0, 1, 2, \ldots$ and converges to ξ. Moreover, the following estimates hold*

$$|y_n - \xi| \leq g_1(|x_n - \xi|)|x_n - \xi| \leq |x_n - \xi| < r, \tag{2.12}$$

$$|z_n - \xi| \leq g_2(|x_n - \xi|)|x_n - \xi| \leq |x_n - \xi| \tag{2.13}$$

and

$$|x_{n+1} - \xi| \leq g_3(|x_n - \xi|)|x_n - \xi| \leq |x_n - \xi|, \tag{2.14}$$

where the "g" functions are defined previously. Furthermore, the limit point ξ is the only solution of equation $F(x) = 0$ in $D_1 := D \cap \bar{U}(\xi, T)$ for $T \in [r, \frac{2}{L_0})$.

Estimates (2.12)–(2.14) shall be shown using mathematical induction. By hypothesis $x_0 \in U(\xi, r) \backslash \{\xi\}$, (2.2) and (2.7), we get that

$$|F'(\xi)^{-1}(F'(x_0) - F'(\xi))| \leq L_0|x_0 - \xi| < L_0 r < 1. \tag{2.15}$$

It follows from estimate (2.15) and the Banach lemma on invertible functions [5, 8, 21, 22, 25] that $F'(x_0) \neq 0$ and

$$|F'(x_0)^{-1}F'(\xi)| \leq \frac{1}{1 - L_0|x_0 - \xi|}. \tag{2.16}$$

Hence, y_0 is well defined. Using (2.3), (2.4), (2.7), (2.9) and the first substep of method (1.2) for $n = 0$, we get in turn that

$$\begin{aligned} |y_0 - \xi| &\le |x_0 - \xi - F'(x_0)^{-1}F'(x_0)|, \\ &\le |F'(x_0)^{-1}F'(\xi)|\,|\int_0^1 F'(\xi)^{-1}[F'(\xi+\theta(x_0-\xi)) - F'(x_0))](x_0-\xi)d\theta|, \\ &\le \frac{L|x_0-\xi|^2}{2(1-L_0|x_0-\xi|)} = g_1(|x_0-\xi|)|x_0-\xi| \le |x_0-\xi| < r, \end{aligned} \tag{2.17}$$

which shows (2.12) for $n = 0$ and $y_0 \in U(\xi, r)$. Notice that $|\xi + \theta(x_0 - \xi) - \xi| \le \theta|x_0 - \xi| < r$ for each $\theta \in [0, 1]$, so $\xi + \theta(x_0 - \xi) \in U(\xi, r)$. We can write by (2.7) that

$$F(x_0) = F(x_0) - F(\xi) = \int_0^1 F'(\xi + \theta(x_0 - \xi))(x_0 - \xi)d\theta. \tag{2.18}$$

and by (2.10)

$$|F'(\xi)^{-1}F(x_0)| \le M|x_0 - \xi|. \tag{2.19}$$

We also get by (2.17) and (2.18) (for $x_0 = y_0$) that since $y_0 \in U(\xi, r)$

$$|F'(\xi)^{-1}F(y_0)| \le M|y_0 - \xi| \le Mg_1(|x_0 - \xi|)|x_0 - \xi|. \tag{2.20}$$

Next, we must show that $A_0 \ne 0$ to define z_0. Using (2.3), (2.5), (2.7), (2.8) and (2.20), we obtain in turn that

$$\begin{aligned} |\big(F'(\xi)(x_0-\xi)\big)^{-1}\big(A_0 - F'(\xi)\big)| &\le |x_0-\xi|^{-1}\Big(|F'(\xi)^{-1}\big(F(x_0) - F'(\xi)(x_0-\xi)\big)| + 2|F'(\xi)^{-1}F(y_0)|\Big) \\ &\le \frac{L_0}{2}|x_0-\xi| + 2g_1(|x_0-\xi|) \\ &= p(|x_0-\xi|) < p(r) < 1, \end{aligned} \tag{2.21}$$

so $A_0 \ne 0$, z_0 is well defined by the second substep of method (1.2) for $n = 0$ and

$$|A_0^{-1}F'(\xi)| \le \frac{1}{|x_0-\xi|(1-p(|x_0-\xi|))}. \tag{2.22}$$

Then, by (2.3), (2.4), (2.16), (2.17), (2.19), (2.20) and (2.22), we get that

$$\begin{aligned} |z_0-\xi| &\le |y_0-\xi| + |A_0^{-1}F'(\xi)||F'(x_0)^{-1}F(y_0)||F'(x_0)^{-1}F'(\xi)||F'(\xi)^{-1}F(x_0)| \\ &\le \Big(1 + \frac{M^2}{|x_0-\xi|(1-p(|x_0-\xi|))(1-L_0|x_0-\xi|)}\Big)|y_0-\xi| \\ &\le g_2(|x_0-\xi|)|x_0-\xi| \le |x_0-\xi| < r, \end{aligned} \tag{2.23}$$

which shows (2.13) for $n = 0$ and $z_0 \in U(\xi, r)$.

We also have that

$$|F'(\xi)^{-1}F(z_0)| \le M|z_0 - \xi| \le Mg_2(|x_0 - \xi|)|x_0 - \xi|, \tag{2.24}$$

since $z_0 \in U(\xi, r)$. By (2.5), (2.16) and (2.24), we obtain in turn that

$$\begin{aligned} |F'(\xi)^{-1}\big([x_0, z_0; F] - F'(\xi)\big)| &\le \frac{L_0}{2}\,(|x_0-\xi| + |z_0-\xi|) \\ &\le \frac{L_0}{2}\,(1 + g_2(|x_0-\xi|))\,|x_0-\xi| \\ &= p_1(|x_0-\xi|) \le p(r) < 1 \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} |F'(\xi)^{-1}\big([y_0,z_0;F]-F'(\xi)\big)| &\le \frac{L_0}{2}\left(|y_0-\xi|+|z_0-\xi|\right) \\ &\le \frac{L_0}{2}\left(g_1(|x_0-\xi|)+g_2(|x_0-\xi|)\right)|x_0-\xi| \\ &= p_2(|x_0-\xi|)<1, \end{aligned} \tag{2.26}$$

so

$$|[x_0,z_0;F]^{-1}F'(\xi)| \le \frac{1}{1-p_1(|x_0-\xi|)} \tag{2.27}$$

and

$$|[y_0,z_0;F]^{-1}F'(\xi)| \le \frac{1}{1-p_2(|x_0-\xi|)}. \tag{2.28}$$

Now, we need an estimate on $|H(t_n)|$. Using (2.1), (2.20), (2.21) and (2.24), we have

$$\begin{aligned} |H(t_n)| &\le |H(|t_n|)| \\ &\le |H\Big(\frac{Mg_2(|x_0-\xi|)|x_0-\xi|}{|x_0-\xi|(1-\frac{L_0}{2}|x_0-\xi|)}\Big)| \\ &\le |H\Big(\frac{Mg_2(|x_0-\xi|)}{1-\frac{L_0}{2}|x_0-\xi|}\Big)|. \end{aligned} \tag{2.29}$$

Hence, x_1 is well defined. Then, using (2.2), (2.5), (2.6), (2.16), (2.25)–(2.29) we get that

$$\begin{aligned} |x_1-\xi| &\le |z_0-\xi|+|H(t_0)||[x_0,z_0;F]^{-1}F'(\xi)||F'(\xi)^{-1}[x_0,z_0;F]||[y_0,z_0;F]^{-1}F'(\xi)||F'(\xi)^{-1}F(z_0)| \\ &\le \left(1+\frac{M^2\big(1+g_1(|x_0-\xi|)\big)H\Big(\frac{Mg_2(|x_0-\xi|)}{1-\frac{L_0}{2}|x_0-\xi|}\Big)|x_0-\xi|}{(1-p_1(|x_0-\xi|))(1-p_2(|x_0-\xi|))}\right)|z_0-\xi| \\ &\le g_3(|x_0-\xi|)|x_0-\xi| \le |x_0-\xi| < r, \end{aligned} \tag{2.30}$$

which implies that (2.14) holds for $n=0$ and $x_1\in U(\xi,r)$. By simply replacing x_0,y_0,z_0,x_1 by x_n,y_n,z_n,x_{n+1} in the preceding estimates, we complete the induction for estimates (2.12)–(2.14). The proof of the uniqueness follows using standard arguments [10].

Remark 2.1 1. *It follows from* (2.8) *that condition* (2.10) *can be dropped, if we set*

$$M(t)=1+L_0t$$

or

$$M(t)=M=2,\ \textit{since}\ t\in\Big[0,\frac{1}{L_0}\Big).$$

2. *The results obtained here can also be used for operators F satisfying autonomous differential equations [5, 8] of the form:*

$$F'(x)=P(F(x)),$$

where P is a continuous operator. Then, since $F'(x^*)=P(F(x^*))=P(0)$, we can apply the results without actually knowing x^*. For example, let $f(x)=e^x-1$. Then, we can choose $P(x)=x+1$.

3. *The radius* $\bar{r_1} = \frac{2}{2L_0+L_1}$ *was shown by Argyros [5] to be the convergence radius of Newton's method*

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \text{ for each } n = 0, 1, 2, \ldots \quad (2.31)$$

under the conditions (2.7)–(2.9) *on* D, *where* L_1 *is the Lipschitz constant on* D. *We have that* $L \leq L_1$ *and* $L_0 \leq L_1$, *so* $\bar{r_1} \leq r_1$. *It follows that the convergence radius* r *of the method* (1.2) *cannot be larger than the convergence radius* r_1 *of the second order Newton's method* (2.31). *As already noted in [5],* $\bar{r_1}$ *is at least as large as the convergence ball given by Rheinboldt [22]*

$$r_R = \frac{2}{3L_1}.$$

In particular, for $L_0 < L_1$, *we have that*

$$r_R < \bar{r_1}$$

and

$$\frac{r_R}{\bar{r_1}} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is our convergence ball $\bar{r_1}$ *is at most three times larger than Rheinboldt's. The same value of* r_R *was given by Traub [25].*

4. *It is worth noticing that method* (1.2) *is not changing when we use the conditions of Theorem 2.1 instead of stronger conditions used in previous studies. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi_1 = \ln\Big(\frac{|x_{n+1}-\xi|}{|x_n-\xi|}\Big) / \ln\Big(\frac{|x_n-\xi|}{|x_{n-1}-\xi|}\Big),$$

or the approximate computational order of convergence (ACOC) defined by

$$\xi_2 = \ln\Big(\frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|}\Big) / \ln\Big(\frac{|x_n-x_{n-1}|}{|x_{n-1}-x_{n-2}|}\Big).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. *Notice also that the computation of* ξ_2 *does not require knowledge of* ξ.

3 Numerical examples

We present numerical examples in this section. Let $H(t) = t$. Then, (2.1) is satisfied if we use $H\Big(\frac{Mg_2(t)}{1-\frac{L_0}{2}t}\Big) = \frac{Mg_2(t)}{1-\frac{L_0}{2}t}$ in all examples that follow.

Example 3.1 *Let* $X = Y = \mathbb{R}$, $D = \bar{U}(0,1)$. *Define* F *on* D *by*

$$F(x) = e^x - 1.$$

Then, $F'(x) = e^x$ *and* $\xi = 0$. *We get that* $L_0 = e - 1 < L = e^{\frac{1}{L_0}} < L_1 = e$ *and* $M = 2$. *Then, for method* (1.2) *the parameters are:*

$$\bar{r_1} = 0.324947, r_1 = 0.382692, r_2 = 0.108641017, r_3 = 0.072182188, r = 0.072182188, \xi_2 = 7.9634.$$

Example 3.2 *Let $D = (-\infty, +\infty)$. Define function F on D by*

$$F(x) = \sin x.$$

Then, we have for $\xi = 0$ that $L_0 = L = L_1 = M = 1$. Then, for method (1.2) *the parameters are:*

$$\bar{r_1} = 0.666667, r_1 = 0.666667, r_2 = 0.305746, r_3 = 0.253071, r = 0.253071, \xi_2 = 11.003.$$

Example 3.3 *Returning back to the motivational example at the introduction of this paper, we have that $L = L_0 = L_1 =$ 146.6629073 and $M = 2$. Then, for method* (1.2) *the parameters are:*

$$\bar{r_1} = 0.00454557, r_1 = 0.00454557, r_2 = 0.001308149, r_3 = 0.00127989798, r = 0.00127989798, \xi_2 = 7.8507.$$

Example 3.4 *Let $X = Y = \mathbb{R}$ and define function F on $D = \mathbb{R}$ by*

$$F(x) = \beta x - \gamma \sin(x) - \delta, \tag{3.1}$$

where β, γ, δ are given real numbers. Suppose that there exists a solution ξ of $F(x) = 0$ with $F'(\xi) \neq 0$. Then, we have

$$L_1 = L_0 = L = \frac{|\gamma|}{|\beta - \gamma \cos \xi|}, \; M = \frac{|\gamma| + |\beta|}{|\beta - \gamma \cos \xi|}.$$

Then one can find the convergence radii for different values of β, γ and δ. As a specific example, let us consider Kepler's equation (3.1) *with $\beta = 1$, $0 \leq \gamma < 1$ and $0 \leq \delta \leq \pi$. A numerical study was presented in [14] for different values of γ and δ.*

Let us take $\gamma = 0.9$ and $\delta = 0.1$. Then the solution is given by $\xi = 0.6308435$. Hence, for method (1.2) *the parameters are:*

$$\bar{r_1} = r_1 = 0.202387, \; r_2 = 0.0108311, \; r_3 = 0.00408931, \; r = 0.00408931, \; \xi_2 = 7.8782.$$

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