International Journal of Advances in Mathematics
Volume 2016, Number 1, Pages 26-36, 2016

Ball convergence for two and three-point methods with memory based on Hermite interpolation

Ioannis K. Argyros[a], Munish Kansal[b,*], V. Kanwar[b]
[a]Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA.
Mail: iargyros@ cameron.edu (Ioannis K. Argyros)
[b]University Institute of Engineering and Technology, Panjab University, Chandigarh-160-014, India.
Mail: mkmaths@ gmail.com (Munish Kansal), vmithil@ yahoo.co.in (V. Kanwar)

ABSTRACT. In this paper, we present a local convergence analysis of multipoint iterative methods with memory for approximating a locally unique solution of a nonlinear equation. The convergence analysis of these type of methods was shown under hypotheses reaching up to the eighth (or even higher) derivative of the function although only the first derivative appears in the method. The main objective of this study is to expand the applicability of these methods using only hypotheses up to the first derivative of the function. In this way, we extend the applicability of these methods under weaker conditions. Furthermore, the radius of convergence and computable error bounds on the distances involved are also included in this study. Finally, numerical examples are presented to show that we can solve equations in cases not possible with earlier approaches.

1 Introduction

Finding rapidly and accurately the zeros of nonlinear functions is an interesting and challenging problem in the field of computational mathematics. In this study, we consider iterative methods for approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

* Corresponding Author: Munish Kansal
Received October 07, 2016; revised November 03, 2016; accepted November 04, 2016.
2010 Mathematics Subject Classification: 65D10, 65D99, 65G99, 47H99.
Key words and phrases: Local convergence; With memory methods; Computational efficiency; R-order of convergence.

where $F : D \subseteq S \rightarrow S$ is a differentiable nonlinear function and D is a convex subset of S ($S = \mathbb{R}$ or $\mathbb{C}$). The problem of finding x^* is very important, since many problems can be reduced to equation (1.1) using mathematical modelling [3, 5, 8, 9, 13, 22, 23, 24, 26]. Analytical methods for solving such equations are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedure. It is well-known that Newton-like methods and its modifications are used for finding the solution of (1.1). Several iterative methods have been developed for solving nonlinear problems by using various techniques such as Taylor's polynomial, decomposition method, homotopy perturbation method, quadrature formulas and many others, see [31, 32, 33, 34] and the references cited therein. For instance, Noor et al. [27] have used the variational iteration technique to develop several iterative methods for solving systems of nonlinear equations. Babolian et al. [28] and Darvishi and Barati [29] have applied Adomian decomposition technique for nonlinear equations. Golbabai and Javidi [30] applied homotopy perturbation method for handling these type of problems. The study about convergence matter of iterative techniques is usually based on two types: semi-local and local convergence analysis.The semi-local convergence matter is, based on the information around an initial guess, to provide conditions ensuring the convergence of the iterative method; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There are lot of papers in the literature which discuss the local and semi-local convergence analysis of Newton-type methods such as [1-34].

Multipoint iterative methods for solving nonlinear equations are of great practical importance since they circumvent the limitations of one-point methods regarding the convergence order and computational efficiency. Generally, multipoint iterative methods are divided into two categories: with memory and without memory methods. The main objective in the construction of the new iterative methods is to obtain the maximum computational efficiency using minimum functional evaluations. According to the Kung-Traub conjecture [26], the order of convergence of any multipoint method without memory requiring n functional evaluations per iteration, cannot exceed the bound 2^{n-1}, called the optimal order.

On the other hand, the basic idea for the construction of multipoint methods with memory was introduced by Traub [26]. He improved a Steffensen-like method by the reuse of information from the previous iteration using secant approach. In fact, he proposed the following method with memory:

$$\begin{cases} \gamma_0 \text{ is given, } \gamma_n = \dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n \geq 1, \\ x_{n+1} = x_n - \dfrac{\gamma_n f(x_n)^2}{f(x_n + \gamma_n f(x_n)) - f(x_n)}, \end{cases} \tag{1.2}$$

having R-order of convergence [21] atleast $1 + \sqrt{2} \approx 2.414$. A similar approach was applied to higher order multipoint methods in [16, 14, 24]. In this study, we present the local convergence analysis of the following iterative method with memory:

$$\begin{cases} y_n = x_n - A_n^{-1} f(x_n), \\ z_n = y_n - B_n^{-1} f(y_n) \left(f(x_n) + \beta f(y_n) \right), \\ x_{n+1} = z_n - \gamma\, C_n^{-1} f(z_n), \quad \gamma \in \mathbb{R}, \end{cases} \tag{1.3}$$

where

$$A_n = f'(x_n) - T_n f(x_n),$$
$$B_n = \left(f'(x_n) - 2T_n f(x_n)\right)\left(f(x_n) + (\beta - 2) f(y_n)\right),$$
$$C_n = [z_n, y_n; f] + [z_n, y_n, x_n; f](z_n - y_n) + [z_n, y_n, x_n, x_n; f](z_n - y_n)(z_n - x_n),$$
$$T_n = \frac{H_2''(x_n)}{2f'(x_n)} = \frac{[x_n, x_n, z_{n-1}; f]}{f'(x_n)},$$

and $[\cdot,\cdot;f]$, $[\cdot,\cdot,\cdot;f]$, $[\cdot,\cdot,\cdot,\cdot;f]$ denote divided differences of order one, two and three, respectively for function f.

Method (1.3) was studied in [20] when $\gamma = 1$. The R-order of convergence of the method (1.3) with memory was found to be nine using Taylor expansions and hypotheses reaching up to the eighth (or even higher) derivatives of the function f. Moreover, in the preceding works the methods were not shown to be well defined or convergent just the order was found assuming that they are well defined and convergent. These problems limit the applicability of these methods.

As a motivational example, define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^4 \ln x^2 + x^6 - x^5, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{1.4}$$

Let $x^* = 1$. We have that

$$f'(x) = 4x^3 \ln x^2 + 6x^5 - 5x^4 + 2x^3,$$
$$f''(x) = 12x^2 \ln x^2 + 30x^4 - 20x^3 + 14x^2,$$
$$f'''(x) = 24x \ln x^2 + 120x^3 - 60x^2 + 52x.$$

Clearly, function $f'''(x)$ is unbounded on D. Hence, the results in [20] cannot be applied to solve equation $f(x) = 0$, where f is given by (1.1). Moreover, the results in [20] do not provide computable convergence radii, error bounds on the distances $|x_n - x^*|$ and uniqueness of the solution results. These results show that if initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^*, the initial guess x_0 should be? These local results give no information on the radius of convergence ball for the corresponding method. We address this question for method (1.3) in Section 2. In the present study, we extend the applicability of these methods by using hypotheses up to the first derivative of function f and contractions. Moreover, we avoid Taylor expansions and use instead Lipschitz parameters. This way we do not have to use higher-order derivatives to show the convergence of these methods. Moreover, we actually show that the method (1.3) is well defined and convergent.

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of method (1.3). The numerical examples are presented in the concluding Section 3.

2 Local convergence

The local convergence analysis of method (1.3) is based on some scalar functions and parameters.
Let $L_0 > 0,\ L > 0,\ M \geq 1,\ K_0 > 0,\ K > 0,\ \beta,\ \gamma \in S$ and $K_1 > 0$ be given parameters. Define functions p_1, $\overline{p_1}$, h_{p_1} and $\overline{h_{p_1}}$ on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} p_1(t) &= L_0 t + \frac{MK_0 t}{1 - L_0 t}, \\ \overline{p_1}(t) &= L_0 t + \frac{2MK_0 t}{1 - L_0 t}, \\ h_{p_1}(t) &= p_1(t) - 1 \end{aligned} \tag{2.1}$$

and

$$\overline{h_{p_1}}(t) = \overline{p_1}(t) - 1. \tag{2.2}$$

We have that $h_{p_1}(0) = -1 < 0$ and $h_{p_1}(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. It then follows from the intermediate value theorem that function h_{p_1} has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_{p_1} and $\overline{r_{p_1}}$ the smallest zeros of functions h_{p_1} and $\overline{h_{p_1}}$, respectively. We have $r_{p_1} < \overline{r_{p_1}}$. Define functions g_1 and h_1 on the interval $[0, r_{p_1})$ by

$$g_1(t) = \Big(\frac{L}{2(1 - L_0 t)} + \frac{M^2 K_0}{(1 - L_0 t)^2 (1 - p_1(t))}\Big) t \tag{2.3}$$

and $h_1(t) = g_1(t) - 1$.

Then, we have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_{p_1}^-$. Denote by r_1 the smallest zero of the function h_1 in the interval $(0, r_{p_1})$. Moreover, define functions p_2, h_{p_2}, $\overline{p_2}$ and $\overline{h_{p_2}}$ on the interval $[0, r_{p_1})$ by

$$p_2(t) = \frac{L_0}{2} t + |\beta - 2| M g_1(t) \tag{2.4}$$

and

$$h_{p_2}(t) = p_2(t) - 1. \tag{2.5}$$

We have $h_{p_2}(0) = -1 < 0$ and $h_{p_2}(t) \to +\infty$ as $t \to r_{p_1}^-$. Denote by r_{p_2} the smallest zero of the function h_{p_2} in the interval $(0, r_{p_1})$. Furthermore, define functions g_2 and h_2 on the interval $[0, r_{p_2})$ by

$$g_2(t) = \left(1 + \frac{M^2 (1 + |\beta| g_1(t))}{(1 - \overline{p_1}(t))(1 - p_2(t))}\right) g_1(t) \tag{2.6}$$

and

$$h_2(t) = g_2(t) - 1. \tag{2.7}$$

We have $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_{p_2}^-$. Denote by r_2 the smallest zero of the function h_2 in the interval $(0, r_{p_2})$.

Now, define functions p_3 and h_{p_3} on the interval $[0, r_{p_2})$ by

$$p_3(t) = \frac{L_0}{2} \left(g_1(t) + g_2(t)\right) t + K \left(g_1(t) + g_2(t)\right) t + K_1 \left(g_1(t) + g_2(t)\right) \left(1 + g_2(t)\right) t^2 \tag{2.8}$$

and

$$h_3(t) = p_3(t) - 1. \tag{2.9}$$

We get in turn that $h_{p_3}(0) = -1 < 0$ and $h_{p_3}(t) \to +\infty$ as $t \to r_{p_2}^-$. Denote by r_{p_3} the smallest zero of the function h_{p_3} in the interval $(0, r_{p_3})$. Finally, define functions g_3 and h_3 on the interval $[0, r_{p_3})$ by

$$g_3(t) = \left(1 + \frac{|\gamma| M}{1 - p_3(t)}\right) g_2(t) \tag{2.10}$$

and

$$h_3(t) = g_3(t) - 1. \tag{2.11}$$

We get that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to r_{p_3}^-$. Denote by r_3 the smallest zero of the function h_3 in the interval $(0, r_{p_3})$.

Define the radius of convergence r by

$$r = min\{r_i\}, \quad i = 1, 2, 3. \tag{2.12}$$

Then, we have that

$$0 < r < \frac{1}{L_0} \tag{2.13}$$

and for each $t \in [0, r)$

$$0 \leq g_i(t) < 1, \quad i = 1, 2, 3, \tag{2.14}$$

and

$$0 \leq p_i(t) < 1, \quad i = 1, 2, 3. \tag{2.15}$$

Let $U(w, \rho)$ and $\overline{U}(w, \rho)$ stand, respectively for the open and closed balls in S with center $w \in S$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.3) using the preceding notation.

Theorem 2.1: Let $f : D \subseteq S \to S$ be a differentiable function and $[\cdot, \cdot; f]$, $[\cdot, \cdot, \cdot; f]$, $[\cdot, \cdot, \cdot, \cdot; f]$ denote divided differences of order one, two and three, respectively for function f on S. Suppose that there exist $x^* \in D$ and $L_0 > 0$ such that for each $x \in D$

$$f(x^*) = 0, \; f'(x^*) \neq 0 \tag{2.16}$$

and

$$|f'(x^*)^{-1}(f'(x) - f'(x^*))| \leq L_0 |x - x^*|. \tag{2.17}$$

Moreover, suppose that there exist $L > 0$, $M \geq 1$, $K_0 > 0$, $K > 0$ and $K_1 > 0$ such that for each $x, y, z \in D_0 := D \cap U(x^*, \frac{1}{L_0})$

$$|f'(x^*)^{-1}(f'(x) - f'(y))| \leq L|x - y|, \tag{2.18}$$

$$|f'(x^*)^{-1} f'(x)| \leq M, \tag{2.19}$$

$$|f'(x^*)^{-1}[x, x, y; f]| \leq K_0, \tag{2.20}$$

$$|f'(x^*)^{-1}[x, y, z; f]| \leq K, \tag{2.21}$$

$$|f'(x^*)^{-1}[x, y, z, z; f]| \leq K_1, \tag{2.22}$$

and

$$\overline{U}(x^*, r) \subseteq D, \tag{2.23}$$

where the radius of convergence r is defined by (2.12). Then, the sequence $\{x_n\}$ generated by method (1.3) for $x_0 \in U(x^*, r)\backslash\{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to the solution x^*. Moreover, the following error estimates hold

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| \leq |x_n - x^*| < r, \tag{2.24}$$

$$|z_n - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| \leq |x_n - x^*| \tag{2.25}$$

and

$$|x_{n+1} - x^*| \leq g_3(|x_n - x^*|)|x_n - x^*|, \tag{2.26}$$

where the "g" functions are defined previously. Furthermore, for $\delta \in [r, \frac{2}{L_0})$, the limit point x^* is the only solution of equation $f(x) = 0$ in $D_1 := D \cap \overline{U}(x^*, \delta)$.

Proof: We shall show that sequence $\{x_n\}$ is well defined and convergent to x^* so that the estimates (2.24)–(2.26) are satisfied using mathematical induction. By hypotheses $x_0 \in U(x^*, r)\backslash\{x^*\}$, (2.13) and (2.17), we have that

$$|f'(x^*)^{-1}(f'(x_0) - f'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1. \tag{2.27}$$

It follows from estimate (2.27) and the Banach lemma on invertible operators [5, 7, 22, 23, 26] that $f'(x_0) \neq 0$ and

$$|f'(x_0)^{-1}f'(x^*)| \leq \frac{1}{1 - L_0|x_0 - x^*|}. \tag{2.28}$$

We shall show that y_0 is well defined from the first substep of method (1.3) by showing $A_0 \neq 0$. Using (2.12), (2.13), (2.15), (2.19), (2.20) and (2.28), we get in turn that

$$\begin{aligned} |f'(x^*)^{-1}\big(F'(x_0) - f'(x^*) - T_0 f(x_0)\big)| &\leq L_0|x_0 - x^*| + |f'(x_0)^{-1}f'(x^*)||f'(x^*)^{-1}f(x_0)||f'(x^*)^{-1}[x_0, x_0, z_{-1}; f]| \\ &\leq L_0|x_0 - x^*| + \frac{MK_0|x_0 - x^*|}{1 - L_0|x_0 - x^*|} = p_1(|x_0 - x^*|) < p_1(r) < 1, \end{aligned} \tag{2.29}$$

so

$$|A_0^{-1}f'(x^*)| \leq \frac{1}{1 - p_1(|x_0 - x^*|)}. \tag{2.30}$$

Using (2.12)–(2.14), (2.16), (2.18), (2.19), (2.20), (2.28) and (2.30), we obtain in turn that

$$\begin{aligned} |y_0 - x^*| &= |x_0 - x^* - f'(x_0)^{-1}f(x_0) + f'(x_0)^{-1}f(x_0) - A_0^{-1}f(x_0)| \\ &\leq |x_0 - x^* - f'(x_0)^{-1}f(x_0)| + |f'(x_0)^{-1}f'(x^*)|^2|f'(x^*)^{-1}f(x_0)|^2|A_0^{-1}f'(x^*)||f'(x^*)^{-1}[x_0, x_0, z_{-1}; f]| \\ &\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{M^2K_0|x_0 - x^*|^2}{(1 - L_0|x_0 - x^*|)^2(1 - p_1|x_0 - x^*|)} \\ &= g_1(|x_0 - x^*|)|x_0 - x^*| \leq |x_0 - x^*| < r, \end{aligned} \tag{2.31}$$

which shows (2.24) for $n = 0$ and $y_0 \in U(x^*, r)$. We shall show the existence of z_0 by proving that $B_0 \neq 0$. Using (2.13), (2.15), (2.17) and (2.31), we get that

$$\begin{aligned}
|\big(f'(x^*)(x_0-x^*)\big)^{-1}\big(f(x_0)-f(x^*)-f'(x^*)(x_0-x^*)+(\beta-2)f(y_0)\big)| &\le |x_0-x^*|^{-1}\Big(\frac{L_0}{2}|x_0-x^*|^2+|\beta-2|M|y_0-x^*|\Big)\\
&\le \frac{L_0}{2}|x_0-x^*|+|\beta-2|Mg_1(|x_0-x^*|)\\
&= p_2(|x_0-x^*|) < p_2(r) < 1,
\end{aligned} \tag{2.32}$$

so

$$|\big(f(x_0)-(\beta-2)f(y_0)\big)^{-1}f'(x^*)| \le \frac{1}{1-p_2|x_0-x^*|}. \tag{2.33}$$

Then, by (2.12)–(2.14), (2.28), (2.30), (2.31) and (2.33), we get that

$$\begin{aligned}
|z_0-x^*| &\le |y_0-x^*| + |\big(f(x_0)+(\beta-2)f(y_0)\big)^{-1}f'(x^*)||A_0^{-1}f'(x^*)||f'(x^*)^{-1}f(y_0)|\Big[|f'(x^*)^{-1}f(x_0)|+|\beta||f'(x^*)^{-1}f(y_0)|\Big]\\
&\le |y_0-x^*| + \frac{M^2|y_0-x^*|\big(|x_0-x^*|+|\beta||y_0-x^*|\big)}{|x_0-x^*|(1-\overline{p_1}(|x_0-x^*|))(1-p_2(|x_0-x^*|))}\\
&\le \Big(1+\frac{M^2\,(1+|\beta|g_1(|x_0-x^*|))\,|x_0-x^*|}{|x_0-x^*|(1-\overline{p_1}(|x_0-x^*|))(1-p_2(|x_0-x^*|))}\Big)g_1(|x_0-x^*|)|x_0-x^*|\\
&= g_2(|x_0-x^*|)|x_0-x^*| \le |x_0-x^*| < r,
\end{aligned} \tag{2.34}$$

which show (2.25) for $n = 0$ and $z_0 \in U(x^*, r)$. Next, we must show that $C_0 \neq 0$ in order to define x_1. In view of (2.12), (2.13), (2.15), (2.17), (2.18), (2.21), (2.22), (2.31) and (2.34), we get in turn that

$$\begin{aligned}
|f'(x^*)^{-1}(C_0-f'(x^*))| &\le |f'(x^*)^{-1}([z_0,y_0;f]-f'(x^*))| + |f'(x^*)^{-1}([z_0,y_0,x_0;f]|\Big(|z_0-x^*|+|y_0-x^*|\Big)\\
&\quad + |f'(x^*)^{-1}([z_0,y_0,x_0,x_0;f])|\Big(|z_0-x^*|+|y_0-x^*|\Big)\Big(|z_0-x^*|+|x_0-x^*|\Big)\\
&\le \frac{L_0}{2}\big(g_1(|x_0-x^*|)+g_2(|x_0-x^*|)\big)|x_0-x^*|\\
&\quad + K\big(g_1(|x_0-x^*|)+g_2(|x_0-x^*|)\big)|x_0-x^*| + K_1\big(g_1(|x_0-x^*|)\\
&\quad + g_2(|x_0-x^*|)\big)\big(1+g_2(|x_0-x^*|)\big)|x_0-x^*|^2\\
&= p_3(|x_0-x^*|) < p_3(r) < 1,
\end{aligned} \tag{2.35}$$

so

$$|C_0^{-1}f'(x^*)| \le \frac{1}{1-p_3(|x_0-x^*|)}. \tag{2.36}$$

Then, by the last substep of method (1.3) for $n = 0$, (2.12)–(2.14), (2.34) and (2.36), we obtain that

$$\begin{aligned}
|x_1-x^*| &\le |z_0-x^*| + |C_0^{-1}f'(x^*)||f'(x^*)^{-1}f(z_0)|\\
&= \left(1+\frac{M|\gamma|}{1-p_3(|x_0-x^*|)}\right)|z_0-x^*|\\
&\le g_3(|x_0-x^*|)|x_0-x^*| \le |x_0-x^*| < r,
\end{aligned} \tag{2.37}$$

which shows (2.26) holds for $n = k$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_n, y_n, z_n, x_{n+1} in the preceding estimates, we complete the induction for estimates (2.24)–(2.26). The proof of the uniqueness follows using standard arguments [10].

Remark 2.1:

1. It follows from (2.17) that condition (2.19) can be dropped, if we set

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2, \text{ since } t \in \Big[0, \frac{1}{L_0}\Big).$$

2. The results obtained here can also be used for operators F satisfying autonomous differential equations [5, 7] of the form:

$$F'(x) = P(F(x)),$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.

3. It is worth noticing that method (1.3) is not changing when we use the conditions of Theorem 2 instead of stronger conditions used in [25]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\Big(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\Big) / \ln\Big(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\Big),$$

or the approximate computational order of convergence (ACOC) defined by

$$\xi_1 = \ln\Big(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\Big) / \ln\Big(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\Big).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. Notice also that the computation of ξ_1 does not require knowledge of x^*.

4. If $\gamma = 0$, we obtain results for method

$$\begin{aligned} y_n &= x_n - A_n^{-1} f(x_n), \\ x_{n+1} &= y_n - B_n^{-1} f(y_n)(f(x_n) + \beta f(y_n)) \end{aligned} \tag{2.38}$$

by replacing z_n by x_{n+1} in method (1.3). Method (2.38) was studied in [20] and convergence order was found to be $\frac{5+\sqrt{17}}{2}$. Moreover, if $\gamma = 1$, we obtain the method studied in [20].

5. If $T_n = T$, then methods (1.3) and (2.38) reduce to the following methods [12]:

$$\begin{aligned} y_n &= x_n - \overline{A_n}^{-1} f(x_n), \\ z_n &= y_n - \overline{B_n}^{-1} f(y_n)(f(x_n) + \beta f(y_n)), \\ x_{n+1} &= z_n - \gamma\, C_2^{-1} f(z_n), \end{aligned} \tag{2.39}$$

where $\overline{A_n} = f'(x_n) - Tf(x_n)$ and $\overline{B_n} = (f'(x_n) - 2Tf(x_n))(f(x_n) + (\beta - 2)f(y_n))$ and

$$\begin{aligned} y_n &= x_n - \overline{A_n}^{-1} f(x_n), \\ z_n &= y_n - \overline{B_n}^{-1} f(y_n)(f(x_n) + \beta f(y_n)), \end{aligned} \tag{2.40}$$

respectively. The functions above Theorem 2 must be redefined by replacing $\frac{K_0}{1-L_0 t}$ by T as follows:

$$p_1(t) = L_0 t + Mt, \tag{2.41}$$

and

$$g_1(t) = \Big(\frac{L}{2(1-L_0 t)} + \frac{M^2}{(1-L_0 t)(1-p_1(t))}\Big)t. \tag{2.42}$$

The rest of the functions remain the same.

6. If f is three times differentiable, we can set

$$K = K_0 = \sup_{x\in D} \frac{|f'(x^*)^{-1} f''(x)|}{2!} \tag{2.43}$$

and

$$K_1 = \sup_{x\in D} \frac{|f'(x^*)^{-1} f'''(x)|}{3!} \tag{2.44}$$

3 Numerical examples

We present numerical examples in this section.

Example 3.1: Let $X = Y = \mathbb{R}$, $D = \overline{U}(0,1)$. Define f on D by

$$f(x) = e^x - 1. \tag{3.1}$$

Then, $f'(x) = e^x$ and $x^* = 0$. We get that $L_0 = e-1 < L = e^{\frac{1}{L_0}}$, $M = e$, $K = K_0 = \frac{e}{2}$ and $K_1 = \frac{e}{6}$. Then, for method (1.3) the parameters are:

$$r_1 = 0.118146, r_2 = 0.0248464,\ r_3 = 0.012325,\ r = 0.012325,\ \xi_1 = 8.9956.$$

Example 3.2: Let $D = (-\infty, +\infty)$. Define function f on D by

$$f(x) = \sin x. \tag{3.2}$$

Then, we have for $x^* = 0$ that $L_0 = L = M = 1$, $K = K_0 = \frac{1}{2}$ and $K_1 = \frac{1}{6}$. Then, for method (1.3) the parameters are:

$$r_1 = 0.38369,\ r_2 = 0.1771,\ r_3 = 0.126501, r = 0.1771,\ \xi = 8.8456.$$

Example 3.3: Returning back to the motivational example at the introduction of this paper, we have that $L = L_0 = \frac{1}{3}\big(4(\frac{5}{2})^3 ln(\frac{5}{2})^2 + 6(\frac{5}{2})^2 + 5(\frac{5}{2})^4 + 2(\frac{5}{2})^3\big)$, $M = 2$, $K = K_0 = \frac{L}{2}$ and $K_1 = \frac{1}{18}\big(24(\frac{5}{2})ln(\frac{5}{2})^2 + 120(\frac{5}{2})^3 + 60(\frac{5}{2})^2 + 52(\frac{5}{2})\big)$. Then, for method (1.3) the parameters are:

$$r_1 = 0.00191409,\ r_2 = 0.000358289,\ r_3 = 0.0001655345, r = 0.0001655345,\ \xi = 8.9823.$$

References

[1] Adomian, G.: Solving Frontier problem of physics: The decomposition method, Kluwer Academic Publishers, Dordrechet, 1994.

[2] Amat, S., Busquier, S., Guttiérrez, J.M.: Geometric constructions of iterative functions to solve nonlinear equations. *J. Comput. Appl. Math.* **157** (2003), 197–205

[3] Amat, S., Busquier, S., Plaza, S.: Dynamics of the King's and Jarratt iterations, Aequationes. Math. **69** (2005), 212–213.

[4] Amat, S., Hernández, M.A., Romero, N.: A modified Chebyshev's iterative method with at least sixth order of convergence. Appl. Math. Comput. **206 (1)** (2008), 164–174

[5] Argyros, I.K.: *Convergence and Applications of Newton-type Iterations.* Springer, (2008).

[6] Argyros, I.K. Chen D., Quian, Q.: The Jarratt method in Banach space setting. J. Comput. Appl. Math. **51** (1994), 103–106.

[7] Argyros, I.K., Hilout S.: *Computational Methods in Nonlinear Analysis.* World Scientific Publ. Comp. New Jersey, (2013)

[8] Argyros, I.K., Magreñán A.A.: Ball convergence theorems and the convergence planes of an iterative method for nonlinear equations. SeMA, **71 (1)** (2015), 39–55.

[9] Argyros, I.K., George, S.: Local convergence of some high-order Newton-like method with frozen derivatives. SeMA. doi: 10.1007/s40324-015-00398-8.

[10] Argyros, I.K., Local convergence for an efficient eighth order iterative method with a parameter for solving equations under weaker conditions. *Int. J. Appl. Comput. Math.* DOI 10.1007/s40819-015-0078-y.

[11] Babolian, E., Biazar, J., Vahidi, A.R.: Solution of a system of nonlinear equations by Adomian decomposition method. Appl. Math. Comput. **150** (2004), 847–854

[12] Behl, R., Kanwar, V.: New highly efficient families of higher-order methods for simple roots, permitting $f'(x_n) = 0$. Int. J. Math and Math. Sc. 2014, 1–12.

[13] Cordero, A., Torregrosa, J.R., Vassileva, M.P.: Increasing the order of convergence of iterative schemes for solving nonlinear systems, J. Comput. Appl. Math. **252** (2012), 86–94

[14] Cordero, A., Lotfi, T., Bakhtiari, P., Torregrosa, J.R.: An efficient two-parametric family with memory for nonlinear equations. Numer. Algorithms. **68**, 323–335 (2015)

[15] Danby, J. M. A., Burkardt, T. M.: The solution of Kepler's equation. I. Celest. Mech. **31** (1983), 95–107

[16] Dzunić, J., Petkovic, M.S.: On generalized multipoint root-solvers with memory. J. Comput. Appl. Math. **236**, 2909–2920 (2012)

[17] Ezquerro, J.A., Hernández, M.A.: New iterations of R-order four with reduced computational cost. BIT Numer Math. **49** (2009), 325–342

[18] Gutiérrez, J.M., Hernández, M.A.: Recurrence relations for the super-Halley method. *Comput. Math. Appl.* **36** (1998), 1–8

[19] Montazeri, H., Soleymani, F., Shateyi, S., Motsa, S.S.: On a new method for computing the numerical solution of systems of nonlinear equations, J. Appl. Math. 2012, Article ID 751975.

[20] Choubey, N., Jaiswal, J.P.: Two and three-point with memory methods for solving nonlinear equations. Appl. Math. Comput. (to appear)

[21] Ortega, J.M., Rheinboldt, W.C.: Iterative Solutions of Nonlinear Equations in Several Variables. Academic Press, New York, (1970)

[22] Potra, F.A.; Pták, V.: Nondiscrete introduction and iterative processes. *Research Notes in Mathematics.* **103** (1984), Pitman, Boston, MA.

[23] Rheinboldt, W.C.: An adaptive continuation process for solving systems of nonlinear equations. Mathematical models and numerical methods (A.N.Tikhonov et al. eds.) **3 (19)**, 129–142 , 1977, Banach center, Warsaw, Poland.

[24] Sharma, J.R., Guha, R.K., Sharma, R.: An efficient fourth-order weighted-Newton method for systems of nonlinear equations, Numer. Algor. **62** (2013), 307–323

[25] Soleymani, F., Lofti, T., Haghani, F.K.: A super-fast tri-parametric iterative method with memory. Appl. Math. Comput. (to appear)

[26] Traub, J.F.: Iterative Methods for the Solution of Equations, Prentice-Hall, New Jersey (1964)

[27] Noor, M.A., Shah, F.A., Noor, K.I., Al-Said, E.: Variational iteration technique for finding multiple roots of nonlinear equations. Sci. Res. Essays **6** (2011), 1344–1350.

[28] Babolian, E., Biazar, J., Vahidi, A.R.: Solution of a system of nonlinear equations by Adimian decomposition method. Appl. Math. Comput. **150** (2004), 847-854.

[29] Darvishi, M.T., Barati, A.: A third-order Newton-type method to solve systems of nonlinear equations, Appl. Math. Comput. **187** (2007), 630-635.

[30] Golbabai, A., Javidi, M.: A new family of iterative methods for solving system of nonlinear algebraic equations. Appl. Math. Comput. **190** (2007), 1717-1722.

[31] Wazwaz, A. M.: A comparison between Adomian decomposition method and Taylor series method in the series solutions. Appl. Math. Comput. **97 (1)** (1998), 37-44.

[32] Bildik, N., Deniz, S.: Comparative study between Optimal Homotopy Asymptotic Method and Perturbation-Iteration Technique for different types of nonlinear equations, Iranian Journal of Science and Technology Transactions A: Science, 1-8, (2016); doi: 10.1007/s40995-016-0039-2

[33] Odibat, Z. M., Momani, S.: Application of variational iteration method to nonlinear differential equations of fractional order. International Journal of Nonlinear Sciences and Numerical Simulation. **7 (1)** (2006), 27–34.

[34] Deniz, S., Bildik, N.: Optimal Perturbation Iteration Method for Bratu-Type Problems. Journal of King Saud University-Science (2016), http://dx.doi.org/10.1016/j.jksus.2016.09.001.