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Generalized fuzzy soft quasi separation axioms in generalized fuzzy soft topological spaces

Fathi. H. Khedr[1] and Mohammed. S. Malfi[1,*]
[1]Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt.
*Department of Mathematics, Faculty of Science, Amran University, Amran, Yemen
*E-Mail: mulfy_s76@yahoo.com

ABSTRACT. In this paper, generalized fuzzy soft quasi separation axioms in generalized fuzzy soft fuzzy topological such as generalized fuzzy soft quasi R_i-spaces $(i = 0, 1)$, generalized fuzzy soft quasi regular (normal)-spaces, and generalized fuzzy soft quasi T_i-spaces $(i = 0, 1, 2, 3, 4)$ are defined and studied by using generalized fuzzy soft quasi-coincident relation and generalized fuzzy soft quasi -neighborhood system. We discuss its characterizations and relationship among them. Also, generalized fuzzy soft hereditary property are discussed.

1. Introduction

L. A. Zadeh [30] in (1965), introduced the concept of fuzzy set and fuzzy set operations. Chang [4] in (1968) introduced the concept of fuzzy topology on a set X by axiomatizing a collection τ of fuzzy subsets of X.

The concept of soft sets was first introduced by Molodtsov [24] in 1999 as a general mathematical tool for dealing with uncertain objects. In [24, 25], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [21], the properties and applications of soft set theory have been studied increasingly [1, 18, 25]. C. Agman et al. [2] and Shabir et al. [28] introduced soft topological space independently in 2011.

* Corresponding Author.
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Maji et al. [20] introduced the concept of fuzzy soft set and some properties regarding fuzzy soft union, intersection, complement of a fuzzy soft set, De Morgan Law etc. Tanay and Kandemir [29] introduced the definition of a fuzzy soft topology over a subset of the initial universe set. Later, Roy and Samanta [27] gave the definition of fuzzy soft topology over the initial universe set. In [11], Kharal and Ahmed defined the notion of a mapping on classes of fuzzy soft sets, which is fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets.

Majumdar and Samanta [23] introduced the notion of generalized fuzzy soft set as a generalization of fuzzy soft sets and studied some of its basic properties. Chakraborty and Mukherjee. [22] gave the topological structure of generalized fuzzy soft sets. Khedr et al. [16] introduced the concept of a generalized fuzzy soft point, a generalized fuzzy soft base (subbase), a generalized fuzzy soft subspace. Khedr et al. [17] introduced the concept of a generalized fuzzy soft mapping on families of generalized fuzzy soft sets. Khedr et al. [12, 13] introduced the generalized fuzzy soft connectedness and generalized fuzzy soft C_i-connectedness ($i = 1, 2, 3, 4$). They introduced different notions of separated and connectedness of generalized fuzzy soft sets and study the relation between these notions.

The concept of separation axioms is one of the most important concepts in topological spaces. In fuzzy setting, it had been studied by many authors such as: Das et al .[5], Saha et al. [6], Hutton et al. [8], and Kandil et al. [9]. In soft setting, it has been studied by Shabir et al .[28] and Gocur et al. [7].

Mahanta and Das [19] introduced the fuzzy soft separation axioms T_i ($i = 0, 1, 2, 3, 4$) by using the definitions of a 'fuzzy soft point' and 'the complement of a fuzzy soft point is a fuzzy soft point ', and 'distinct of fuzzy soft points' in his sense. Khedr et al. [14] introduced fuzzy soft separation axioms in terms of the modified definitions of a 'fuzzy soft point', the complement of a fuzzy soft point is a fuzzy soft set' and 'distinct of fuzzy soft points' [23] and we study some of their properties. Kandil et al .[10] introduced the notion of separation axioms in fuzzy soft topological spaces by using quasi-coincident relation and fuzzy soft neighborhood system.

The notion of separation axioms in generalized fuzzy topological spaces has been studied by Khedr et al. [15] by using generalized fuzzy soft open sets and generalized fuzzy soft neighborhood system. The object of the present paper is to introduce a set of new generalized fuzzy soft regularity and generalized fuzzy soft separation axioms which are called generalized fuzzy soft quasi R_i-spaces ($i = 0, 1$), generalized fuzzy soft quasi regular (normal)-spaces, and generalized fuzzy soft quasi T_i-spaces ($i = 0, 1, 2, 3, 4$) by using generalized fuzzy soft quasi-coincident and generalized fuzzy soft quasi$-$neighborhood system.

2. Preliminaries

In this section, we will give some fundamental definitions and theorems about generalized fuzzy soft sets, generalized fuzzy soft topology and generalized fuzzy soft continuous mappings which will be needed in the sequel.

Definition 2.1. [30] Let X be a non-empty set. A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0, 1]$ whose value $\mu_A(x)$ represents the 'grade of membership' of x in A for $x \in X$. The set of all fuzzy sets in a set

X is denotes by I^X, where I is the closed unit interval $[0,1]$

Definition 2.2. [27] Let X be an initial universe set and E be a set of parameters. Let $A \subseteq E$. A fuzzy soft set f_A over X is a mapping from E to I^X, i.e., $f_A : E \longrightarrow I^X$, where $f_A(e) \neq \bar{0}$ if $e \in A \subset E$, and $f_A(e) = \bar{0}$ if $e \notin A$, where $\bar{0}$ is denotes empty fuzzy set in X.

Definition 2.3. [22] Let X be a universal set of elements and E be a universal set of parameters for X. Let $F : E \longrightarrow I^X$ and μ be a fuzzy subset of E, i.e., $\mu : E \longrightarrow I$. Let F_μ be the mapping $F_\mu : E \longrightarrow I^X \times I$ defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I$. Then F_μ is called a generalised fuzzy soft set ($GFSS$ in short) over (X, E). The family of all generalized fuzzy soft sets ($GFSSs$ in short) over (X, E) is denotes by $GFSS(X, E)$.

Definition 2.4. [22] Let F_μ and G_δ be two $GFSSs$ over (X, E). F_μ is said to be a GFS subset of G_δ, denoted by $F_\mu \sqsubseteq G_\delta$, if

(1) μ is a fuzzy subset of δ;

(2) $F(e)$ is also a fuzzy subset of $G(e)$, $\forall e \in E$.

Definition 2.5. [22] Let F_μ be a $GFSS$ over (X, E). The generalized fuzzy soft complement of F_μ, denoted by F^c_μ, is defined by F^c_μ=G_δ, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e)\ \forall e \in E$. Obviously ($F^c_\mu)^c$=F_μ.

Definition 2.6. [3] Let F_μ and G_δ be two $GFSSs$ over (X, E). The union of F_μ and G_δ, denoted by $F_\mu \sqcup G_\delta$, is The $GFSSH_\nu$, defined as $H_\nu : E \longrightarrow I^X \times I$ such that $H_\nu(e) = (H(e), \nu(e))$, where $H(e) = F(e) \vee G(e)$ and $\nu(e) = \mu(e) \vee \delta(e), \forall e \in E$.

Definition 2.7. [3] Let F_μ and G_δ be two $GFSSs$ over (X, E). The Intersection of F_μ and G_δ, denoted by $F_\mu \sqcap G_\delta$, is the $GFSS\ M_\sigma$, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that $M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = F(e) \wedge G(e)$ and $\sigma(e) = \mu(e) \wedge \delta(e), \forall e \in E$.

Definition 2.8. [22] A$GFSS$ is said to be a generalized null fuzzy soft set, denoted by $\widetilde{0}_\theta$, if $\widetilde{0}_\theta : E \longrightarrow I^X \times I$ such that $\widetilde{0}_\theta(e) = (\widetilde{0}(e), \theta(e))$ where $\widetilde{0}(e) = \overline{0}\ \ \forall e \in E$ and $\theta(e) = 0\ \forall e \in E$ (Where $\overline{0}(x) = 0$, $\forall x \in X$).

Definition 2.9. [22] A $GFSS$ is said to be a generalized absolute fuzzy soft set, denoted by $\widetilde{1}_\triangle$, if $\widetilde{1}_\triangle : E \longrightarrow I^X \times I$, where $\widetilde{1}_\triangle(e) = (\widetilde{1}(e), \triangle(e))$ is defined by $\widetilde{1}(e) = \overline{1}$, $\forall\ e \in E$ and $\triangle(e) = 1$, $\forall e \in E$ (Where $\overline{1}(x) = 1, \forall x \in X$).

Definition 2.10. [3] Let T be a collection of generalized fuzzy soft sets over (X, E). Then T is said to be a generalized fuzzy soft topology ($GFST$, in short) over (X, E) if the following conditions are satisfied:

(1) $\widetilde{0}_\theta$ and $\widetilde{1}_\triangle$ are in T.

(2) Arbitrary unions of members of T belong to T.

(3) Finite intersections of members of T belong to T.

The triple (X, T, E) is called a generalized fuzzy soft topological space ($GFST$- space, in short) over (X, E).

The member of T are called generalized fuzzy soft open set [GFS open for short] in (X, T, E) and their generalized fuzzy soft complements are called GFS closed sets in (X, T, E). The family of all GFSclosed sets in (X, T, E) is denotes by T^c.

Definition 2.11. [3] Let (X, T, E) be a $GFST-$space and $F\mu \in GFSS(X, E)$. The generalized fuzzy soft closure of $F\mu$, denoted by $cl(F\mu)$, is the intersection of all GFS closed supper sets of $F\mu$. i.e., $cl(F\mu) = \sqcap\{H_\nu : H_\nu \in T^c, F\mu \sqsubseteq H_\nu\}$. Clearly, $cl(F\mu)$ is the smallest GFS closed set over (X, E) which contains $F\mu$.

Definition 2.12. [16] The generalized fuzzy soft set $F_\mu \in GFSS(X, E)$ is called a generalized fuzzy soft point (GFS point for short) over (X, E) if there exist $e \in E$ and $x \in X$ such that

(1) $F(e)(x) = \alpha(0 < \alpha \leq 1)$ and $F(e)(y) = 0$ for all $y \in X - \{x\}$,

(2) $\mu(e) = \lambda(0 < \lambda \leq 1)$ and $\mu(e') = 0$ for all $e' \in E - \{e\}$. We denote this generalized fuzzy soft point $F_\mu = (e_\lambda, x_\alpha)$.

(e, x) and (λ, α) are called respectively, the support and the value of (e_λ, x_α). The class of all GFS points in (X, E), denoted by $GFSP(X, E)$.

Definition 2.13. [16] Let F_μ be a $GFSS$ over (X, E). We say that $(e_\lambda, x_\alpha)\widetilde{\in} F_\mu$ read as (e_λ, x_α) belongs to the $GFSSF_\mu$ if for the element $e \in E$, $\alpha \leq F(e)(x)$ and $\lambda \leq \mu(e)$.

Definition 2.14. [26] Let $F_\mu, G_\delta \in GFSS(X, E)$ over (X, E). F_μ is said to be a generalised soft quasi-coincident with [GFS quasi-coincident in short] G_δ, denoted by $F_\mu q G_\delta$, if there exist $e \in E$ and $x \in X$ such that $F(e)(x) + G(e)(x) > 1$ and $\mu(e) + \delta(e) > 1$.

If F_μ is not generalised soft quasi-coincident with G_δ, then we write $F_\mu \bar{q} G_\delta$ i.e, For every $e \in E$ and $x \in X$, $F(e)(x) + G(e)(x) \leq 1$ or for every $e \in E$ and $x \in X$, $\mu(e) + \delta(e) \leq 1$.

Definition 2.15. [26] Let (x_α, e_λ) be a generalized fuzzy soft point and F_μ be a $GFSS$ over (X, E). (x_α, e_λ) is said to be generalised soft quasi-coincident with F_μ, denoted by $(x_\alpha, e_\lambda) q F_\mu$, if and only if there exists an element $e \in E$ such that $\alpha + F(e)(x) > 1$ and $\lambda + \mu(e) > 1$.

Definition 2.16. [12] Let $F_\mu \in GFSS(X, E)$. The generalized fuzzy soft support (GFS support in short) of F_μ defined by $S(F_\mu)$ is the set, $S(F_\mu) = \{x \in X, e \in E : F(e)(x) > 0 \text{ and } \mu(e) > 0\}$.

Definition 2.17. [17] Let (X, T, E) be a $GFST-$space and $F\mu, H_\nu$ $GFSSs$ in $GFS(X, E)$. F_μ is called generalized fuzzy soft $Q-$neighborhood (briefly, GFS $Q-$nbd) of H_ν if there exists $G_\delta \in T$ such that $H_\nu q G_\delta$ and $G_\delta \sqsubseteq F_\mu$. The family of all GFS $Q-$nbds of H_ν, denoted by $N_q(H_\nu)$.

Definition 2.18. [17] Let (X, T, E) be a $GFST-$space and (e_λ, x_α) be a GFS point in (X, E). A $GFSS$ F_μ is called GFS $Q-$nbd of (e_λ, x_α), if there exists $G_\delta \in T$ such that $(e_\lambda, x_\alpha) q G_\delta$ and $G_\delta \sqsubseteq F_\mu$. The family of all GFS $Q-$nbds of (e_λ, x_α), denoted by $N_q(e_\lambda, x_\alpha)$.

Remark 2.19. [17] *If F_μ is GFS open set, then F_μ is a GFS $Q-$nbd of (e_λ, x_α) if and only if $(e_\lambda, x_\alpha) q F_\mu$.*

Theorem 2.20. [26] *Let $F_\mu, G_\delta \in GFSS(X, E)$ and $(e_\lambda, x_\alpha) \in GFSP(X, E)$. Then:*

(1) $F_\mu \bar{q} G_\delta \Leftrightarrow F_\mu \sqsubseteq G^c_\delta$,

(2) $F_\mu q G_\delta \Rightarrow F_\mu \sqcap G_\delta \neq \widetilde{0}_\theta$,

(3) $F_\mu \bar{q} F^c_\mu$,

(4) $(x_\alpha, e_\lambda) \bar{q} F_\mu \Leftrightarrow (x_\alpha, e_\lambda) \widetilde{\in} F^c_\mu$.

Definition 2.21. [12] Let (X, T, E) be a $GFST-$space over (X, E) and G_δ be GFS subset of (X, E). Then $T_{G_\delta} = \{G_\delta \sqcap F_\mu : F_\mu \in T\}$ is called a GFS relative topology and $(G_\delta, T_{G_\delta}, E)$ is called a GFS subspace of (X, T, E). If $G_\delta \in T$(resp, $G_\delta \in T^c$) then $(G_\delta, T_{G_\delta}, E)$ is called generalized fuzzy soft open (resp. closed) subspace of (X, T, E).

3. Generalized fuzzy soft quasi R_i-spaces, $i = 0, 1$ and Generalized fuzzy soft quasi regular (normal)$-$spaces

In this section, we introduce the notion of generalized fuzzy soft quasi R_i-spaces ($GFS\ Q-R_i, i = 0, 1$ for short) and generalized fuzzy soft quasi regular (normal)-space ($GFS\ Q$ regular (normal) space for short).

Proposition 3.1. *Let $N_q(e_\lambda, x_\alpha)$ be the family of all GFS $Q-$nbds of (e_λ, x_α) in a $GFST-$space (X, T, E). The following hold:*

(1) If $F_\mu \in N_q(e_\lambda, x_\alpha)$, then $(e_\lambda, x_\alpha)qF_\mu$,

(2) If $F_\mu \in N_q(e_\lambda, x_\alpha)$ and $F_\mu \sqsubseteq G_\delta$, then $G_\delta \in N_q(e_\lambda, x_\alpha)$,

(3) If $F_\mu, G_\delta \in N_q(e_\lambda, x_\alpha)$, then $F_\mu \sqcap G_\delta \in N_q(e_\lambda, x_\alpha)$,

(4) If $F_\mu \in N_q(e_\lambda, x_\alpha)$ then there exist $G_\delta \in N_q(e_\lambda, x_\alpha)$ such that $G_\delta \sqsubseteq F_\mu$ and $G_\delta \in N_q(e'_\gamma, y_\beta)$ for every GFS point (e'_γ, y_β) which is GFS quasi-coincident with G_δ.

Proof. (1) Suppose $F_\mu \in N_q(e_\lambda, x_\alpha)$. Then there exist $G_\delta \in T$ such that $(e_\lambda, x_\alpha)qG_\delta$ and $G_\delta \sqsubseteq F_\mu$ i.e., $\alpha + G(e)(x) > 1$ and $\lambda + \delta(e) > 1$ for some $x \in X$,$e \in E$. Again $G(e)(x) \leq F(e)(x)$ and $\delta(e) \leq \mu(e)$ for all $x \in X$,$e \in E$. Therefore, $\alpha + F(e)(x) \geq \alpha + G(e)(x) > 1$ and $\lambda + \mu(e) \geq \lambda + \delta(e) > 1$. Hence, (e_λ, x_α) is quasi-coincident with F_μ i.e., $(e_\lambda, x_\alpha)qF_\mu$.

(2) Obvious.

(3) Suppose $F_\mu, G_\delta \in N_q(e_\lambda, x_\alpha)$. Then there exist $H_\nu, K_\gamma \in T$ such that $(e_\lambda, x_\alpha)qH_\nu$, $(e_\lambda, x_\alpha)qK_\gamma$ and $H_\nu \sqsubseteq F_\mu, K_\gamma \sqsubseteq G_\delta$. So, $\alpha + H(e)(x) > 1, \lambda + \nu(e) > 1$ and $\alpha + K(e)(x) > 1, \lambda + \gamma(e) > 1$. Therefore, $\alpha + min\{H(e)(x), K(e)(x)\} > 1$ and $\lambda + min\{\nu(e), \gamma(e)\} > 1$. Also $H_\nu \sqcap K_\gamma \in T$, $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu \sqcap G_\delta$. Then $(e_\lambda, x_\alpha)qH_\nu \sqcap K_\gamma$. Hence $F_\mu \sqcap G_\delta \in N_q(e_\lambda, x_\alpha)$.

(4) Suppose $F_\mu \in N_q(e_\lambda, x_\alpha)$. Then there exists $G_\delta \in T$ such that $(e_\lambda, x_\alpha)qG_\delta$ and $G_\delta \sqsubseteq F_\mu$. So we have, $G_\delta \in N_q(e_\lambda, x_\alpha)$ such that $(e_\lambda, x_\alpha)qG_\delta$ and $G_\delta \sqsubseteq F_\mu$. Let (e'_γ, y_β) any GFS point which is GFS quasi-coincident with G_δ. Therefore $G_\delta \in N_q(e'_\gamma, y_\beta)$. □

The proof of the following theorem follows directly from the definition of GFS set and GFS point and therefore omitted.

Proposition 3.2. *Let $F_\mu, G_\delta, H_\nu \in GFSS(X, E)$ and $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$. Then:*

(1) $F_\mu \bar{q} G_\delta \Longleftrightarrow G_\delta \bar{q} F_\mu$.

(2) $F_\mu \sqcap G_\delta = \widetilde{0}_\theta \Longrightarrow F_\mu \bar{q} G_\delta$.

(3) $F_\mu \bar{q} G_\delta, H_\nu \sqsubseteq G_\delta \Longrightarrow F_\mu \bar{q} H_\nu [F_\mu q H_\nu \Longrightarrow F_\mu q G_\delta$, then $H_\nu \sqsubseteq G_\delta]$.

(4) $F_\mu q G_\delta \Longleftrightarrow$ there exists an $(e_\lambda, x_\alpha) \widetilde{\in} F_\mu$ such that $(e_\lambda, x_\alpha)qG_\delta$.

(5)$F_\mu \sqsubseteq G_\delta \Longleftrightarrow [(e_\lambda, x_\alpha)qF_\mu \Longrightarrow (e_\lambda, x_\alpha)qG_\delta]$ or $[(e_\lambda, x_\alpha)\bar{q}G_\delta \Longrightarrow (e_\lambda, x_\alpha)\bar{q}F_\mu]$.

(6) $(e_\lambda, x_\alpha)q \sqcup_{i\in J} (F_\mu)_i \Longleftrightarrow (e_\lambda, x_\alpha)q(F_\mu)_{i_0}$ for some $i_0 \in J$.

(7) $(e_\lambda, x_\alpha)q(F_\mu \sqcap G_\delta) \Longleftrightarrow [(e_\lambda, x_\alpha)qF_\mu and (e_\lambda, x_\alpha)qG_\delta]$.

(8) $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta) \Longleftrightarrow (x \neq y, e \neq e')$ *or* $(x = y, e = e',$ *but* $\alpha + \beta \leq 1$ *or* $\lambda + \gamma \leq 1)$ *or* $(x = y, e \neq e', \alpha + \beta > 1)$ *or* $(x \neq y, e = e', \lambda + \gamma > 1)$.

(9) $x \neq y$ *or* $e \neq e' \Longrightarrow (e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)\ \forall \alpha, \beta, \lambda, \gamma \in I$ *and* $\forall e, e' \in E$.

Proposition 3.3. *Let (X, T, E) be a $GFST-$space, $F_\mu \in GFSS(X, E)$ and $(e_\lambda, x_\alpha) \in GFSP(X, E)$. Then we have:*

(1) $(e_\lambda, x_\alpha)qcl(F_\mu) \Longleftrightarrow O_{(e_\lambda, x_\alpha)}qF_\mu \forall O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$.

(2) $G_\delta qF_\mu \Longleftrightarrow G_\delta qcl(F_\mu) \forall G_\delta \in T$.

Proof. (1) $(e_\lambda, x_\alpha)qcl(F_\mu)$ if and only if for every GFS closed set H_ν containing F_μ, $(e_\lambda, x_\alpha)qH_\nu$ i.e., $\alpha + H(e)(x) > 1$, $\lambda + \nu(e) > 1$.

That is, $(e_\lambda, x_\alpha)qcl(F_\mu)$ if and only if $1 - H(e)(x) < \alpha$, $1 - \nu(e) < \lambda\ \forall x \in X, \forall e \in E$ and for all GFS closed set $H_\nu \sqsupseteq F_\mu$.

Therefore, $(e_\lambda, x_\alpha)qcl(F_\mu)$ if and only if for every GFS open set $G_\delta \sqsubseteq F^c_\mu$ we have, $G(e)(x) < \alpha, \delta(e) < \lambda$.

In other words, for every GFS open set G_δ satisfying $G(e)(x) \geq \alpha$, $\delta(e) \geq \lambda$ for some $x \in X, e \in E$ is not contained in F^c_μ and is a GFS $Q-$nbd of (e_λ, x_α). Again G_δ is not contained in F^c_μ if and only if G_δ is a GFS quasi-coincident with F_μ i.e., $G_\delta qF_\mu$. Take $O_{(e_\lambda, x_\alpha)} = G_\delta$, then $O_{(e_\lambda, x_\alpha)}qF_\mu$.

(2) It similar to the proof of (1). □

Definition 3.4. A $GFST-$space (X, T, E) is said to be:

(1) generalized fuzzy soft quasi R_0-space ($GFS\ Q - R_0-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta) \Longrightarrow cl(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$.

(2) generalized fuzzy soft quasi R_1-space ($GFS\ Q - R_0-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$ implies $\exists O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$ and $O_{(e'_\gamma, y_\beta)} \in N_q(e'_\gamma, y_\beta)$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{(e'_\gamma, y_\beta)}$.

(3) generalized fuzzy soft quasi regular$-$space ($GFS\ Q\ regular-$space for short) if for every $(e_\lambda, x_\alpha) \in GFSP(X, E)$ and $G_\delta \in T^c$ with $(e_\lambda, x_\alpha)\bar{q}G_\delta$ implies $\exists O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$ and $O_{G_\delta} \in N_q(G_\delta)$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{G_\delta}$.

(4) generalized fuzzy soft quasi normal$-$space ($GFS\ Q\ normal-$space for short) if for every $F_\mu, G_\delta \in T^c$ with $F_\mu\bar{q}G_\delta$ implies $\exists O_{F_\mu} \in N_q(F_\mu)$ and $O_{G_\delta} \in N_q(G_\delta)$ such that $O_{F_\mu}\bar{q}O_{G_\delta}$.

Theorem 3.5. *Let (X, T, E) be a $GFST-$space, $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ and $F_\mu \in T^c$. The following statements are equivalent:*

(1) (X, T, E) *is* $GFSQ - R_0-$*space,*

(2) $(e_\lambda, x_\alpha)qcl(e'_\gamma, y_\beta) \Longrightarrow (e_\lambda, x_\alpha)q(e'_\gamma, y_\beta)$,

(3) $cl(e_\lambda, x_\alpha) \sqsubseteq O_{(e_\lambda, x_\alpha)}$ *for every* $O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$,

(4) $cl(e_\lambda, x_\alpha) \sqsubseteq \sqcap\{O_{(e_\lambda, x_\alpha)} : O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)\}$,

(5) $(e_\lambda, x_\alpha)\bar{q}F_\mu$ *implies there exists* $O_{F_\mu} \in N_q(F_\mu)$ *such that* $(e_\lambda, x_\alpha)\bar{q}O_{F_\mu}$,

(6) $(e_\lambda, x_\alpha)\bar{q}F_\mu$ *implies* $cl(e_\lambda, x_\alpha)\bar{q}F_\mu$,

(7) $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ *implies* $cl(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$.

Proof. (1) $\Longrightarrow$ (2): Let $(e_\lambda, x_\alpha) q cl(e'_\gamma, y_\beta)$. Suppose $cl(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$. Since (X, T, E) is $GFS\ Q - R_0-$space, then $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$ which is a contradicton. Therefore, $cl(e_\lambda, x_\alpha) q (e'_\gamma, y_\beta)$.

(2) $\Longrightarrow$ (3): Let $O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$. Then there exists $G_\delta \in T$ such that $(e_\lambda, x_\alpha) q G_\delta, G_\delta \sqsubseteq O_{(e_\lambda, x_\alpha)}$. By (2) of Proposition 3.3, we have $cl(e_\lambda, x_\alpha) q G_\delta$. By (2), then $(e_\lambda, x_\alpha) q cl(G_\delta)$. By (1) of Proposition 3.3, we have $O_{(e_\lambda, x_\alpha)} q G_\delta)$. Therefore, $cl(e_\lambda, x_\alpha) \sqsubseteq O_{(e_\lambda, x_\alpha)}$ (by 5 of Proposition 3.2).

(3) $\Longrightarrow$ (4): Obvious.

(4) $\Longrightarrow$ (5): Let $(e_\lambda, x_\alpha)\bar{q}F_\mu$. Then $(e'_\gamma, y_\beta)\widetilde{\in}F^c_\mu$. By (3), $cl(e_\lambda, x_\alpha) \sqsubseteq F^c_\mu$ and so $F_\mu \sqsubseteq [cl(e_\lambda, x_\alpha)]^c = O_{F_\mu}$. Thus $(e_\lambda, x_\alpha)\bar{q}[cl(e_\lambda, x_\alpha)]^c = O_{F_\mu}$.

(5) $\Longrightarrow$ (6): Let $(e_\lambda, x_\alpha)\bar{q}F_\mu$. By (5), there exists O_{F_μ} such that $(e_\lambda, x_\alpha)\bar{q}O_{F_\mu}$. Then $(e'_\gamma, y_\beta)\widetilde{\in}O_{F^c_\mu}$ and so $cl(e_\lambda, x_\alpha) \sqsubseteq O_{F^c_\mu}$. Therefore, $cl(e_\lambda, x_\alpha)\bar{q}O_{F_\mu}$. By (1), of Proposition 3.3, $cl(e_\lambda, x_\alpha)\bar{q}F_\mu$.

(6) $\Longrightarrow$ (7) and (7) $\Longrightarrow$ (1) are obvious. ☐

Theorem 3.6. *The following implications holds:*

$$GFS\ Q\ normal \wedge GFS\ Q - R_0 \Longrightarrow GFS\ Q\ regular \Longrightarrow GFS\ Q - R_1 \Longrightarrow GFS\ Q - R_0.$$

Proof. (i) $GFS\ Q$ normal $\wedge GFS\ Q - R_0 \Longrightarrow GFS\ Q$ regula : Let (X, T, E) be a $GFS\ Q\ normal \wedge GFS\ Q - R_0$ space and $(e_\lambda, x_\alpha)\bar{q}F_\mu$ such that $F_\mu \in T^c$. By (6) of Theorem 3.5, we have $cl(e_\lambda, x_\alpha)\bar{q}F_\mu$. Since (X, T, E) is $GFS\ Q$ normal space, then here exist $O_{cl(e_\lambda, x_\alpha)}$ and O_{F_μ} such that $O_{cl(e_\lambda, x_\alpha)}\bar{q}O_{F_\mu}$. Take $O_{(e_\lambda, x_\alpha)} = O_{cl(e_\lambda, x_\alpha)}$, then $O_{(e_\lambda, x_\alpha)}\bar{q}F_\mu$ and hence (X, T, E) is a $GFS\ Q$ regular space.

(ii) $GFS\ Q$ regular $\Longrightarrow GFS\ Q - R_1$: Let (X, T, E) be a $GFS\ Q$ regular space and $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$. Then there exist $O_{(e_\lambda, x_\alpha)}$ and $O_{(e'_\gamma, y_\beta)} \in T$ such that $O_{(e_\lambda, x_\alpha)}\bar{q}O_{(cl(e'_\gamma, y_\beta)}$. Take $O_{(e'_\gamma, y_\beta)} = O_{cl(e'_\gamma, y_\beta)}$, then $O_{(e_\lambda, x_\alpha)}\bar{q}O_{(e'_\gamma, y_\beta)}$ and hence (X, T, E) is a $GFS\ Q - R_1-$space.

(iii) $GFS\ Q - R_1 \Longrightarrow GFS\ Q - R_0$: Let (X, T, E) be a $GFS\ Q - R_1-$space and $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$. Then there exist $O_{(e_\lambda, x_\alpha)}$ and $O_{(e'_\gamma, y_\beta)} \in T$ such that $(e_\lambda, x_\alpha)\bar{q}O_{(e'_\gamma, y_\beta)}$. By (4) of Proposition 3.3, we have $(e'_\gamma, y_\beta)\bar{q}cl(e_\lambda, x_\alpha)$. Hence (X, T, E) is a $GFS\ Q - R_0-$space. ☐

Corollary 3.7. *Let (X, T, E) be a $GFST-$space. Then (X, T, E) is a GFS Q regular space if and only if $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$ implies there exist $O_{cl(e_\lambda, x_\alpha)}$ and $O_{cl(e'_\gamma, y_\beta)} \in T$ such that $O_{cl(e_\lambda, x_\alpha)}\bar{q}O_{cl(e'_\gamma, y_\beta)}$.*

Proof. Follows directly from Theorem 3.6 and Therorem 3.5 (3). ☐

Theorem 3.8. *Let (X, T, E) be a $GFST-$space. Then (X, T, E) is a GFS Q regular space if and only if for all $(e_\lambda, x_\alpha) \in GFSP(X, E)$ and for all $O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$, there exist $O^*_{(e_\lambda, x_\alpha)}$ such that $cl(O^*_{(e_\lambda, x_\alpha)}) \sqsubseteq O_{(e_\lambda, x_\alpha)}$.*

Proof. Let (X, T, E) be a $GFS\ Q$ regular space, $(e_\lambda, x_\alpha) \in GFSP(X, E)$ and $O_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$. Then $(e_\lambda, x_\alpha)\bar{q}O^c_{(e_\lambda, x_\alpha)}$. Therefore, there exists $O^*_{(e_\lambda, x_\alpha)} \in N_q(e_\lambda, x_\alpha)$ and $G_\delta \in N_q(O^c_{(e_\lambda, x_\alpha)})$, such that $O^*_{(e_\lambda, x_\alpha)}\bar{q}G_\delta$. This implies that $O^*_{(e_\lambda, x_\alpha)} \sqsubseteq G^c_\delta \in T^c$. Thus $cl(O^*_{(e_\lambda, x_\alpha)}) \sqsubseteq G^c_\delta \sqsubseteq O_{(e_\lambda, x_\alpha)}$.

Conversely, let $(e_\lambda, x_\alpha) \in GFSP(X, E)$ and $G_\delta \in T^c$ such that $(e_\lambda, x_\alpha)\bar{q}G_\delta$. Then $(e_\lambda, x_\alpha)\widetilde{\in}G^c_\delta$ i.e., $G^c_\delta \in$

$N_q(e_\lambda, x_\alpha)$, so there exists $O^*_{(e_\lambda,x_\alpha)}$ such that $cl(O^*_{(e_\lambda,x_\alpha)}) \sqsubseteq G^c_\delta$. Thus $G_\delta \sqsubseteq [cl(O^*_{(e_\lambda,x_\alpha)})]^c$. Take $O_{G_\delta} = [cl(O^*_{(e_\lambda,x_\alpha)})]^c$. Therefore, $O_{G_\delta}\bar{q}O^*_{(e_\lambda,x_\alpha)}$. Hence, (X, T, E) is a $GFS\ Q$ regular space. □

Theorem 3.9. *A $GFST-$space (X, T, E) is a $GFS\ Q$ regular space if and only if for every $(e_\lambda, x_\alpha) \in GFSP(X, E)$, $F_\mu \in T^c$ with $(e_\lambda, x_\alpha)\bar{q}F_\mu$ implies there exist $O_{(e_\lambda,x_\alpha)}, O_{F_\mu} \in T$ such that $cl(O_{(e_\lambda,x_\alpha)})\bar{q}cl(O_{F_\mu})$.*

Proof. Let (X, T, E) be a $GFS\ Q$ regular space, $(e_\lambda, x_\alpha) \in GFSP(X, E)$ and $F_\mu \in T^c$ with $(e_\lambda, x_\alpha)\bar{q}F_\mu$. Then there exist, $O^{\circledast}_{(e_\lambda,x_\alpha)}, O_{F_\mu} \in T$ such that $O_{F_\mu}\bar{q}O^{\circledast}_{(e_\lambda,x_\alpha)}$ implies that $cl(O_{F_\mu})\bar{q}O^{\circledast}_{(e_\lambda,x_\alpha)}$ (by 2) of Proposition 3.3) that is,

$cl(O_{F_\mu})\bar{q}(e_\lambda, x_\alpha)$. Again, since (X, T, E) is a $GFS\ Q$ regular space, then there exist $O^*_{(e_\lambda,x_\alpha)}, O_{cl(O_{F_\mu})} \in T$ such that $O^*_{(e_\lambda,x_\alpha)}\bar{q}O_{cl(O_{F_\mu})} \Longrightarrow cl(O^*_{(e_\lambda,x_\alpha)})\bar{q}O_{cl(O_{F_\mu})}$ (by 2) of Proposition 3.3). Since (X, T, E) is a $GFS\ Q$ regular space and $O^{\circledast}_{(e_\lambda,x_\alpha)} \in T$, then by the above theorem, there exists $O_{(e_\lambda,x_\alpha)} \in T$ such that $cl(O_{(e_\lambda,x_\alpha)}) \sqsubseteq O^{\circledast}_{(e_\lambda,x_\alpha)}$. Since $cl(O_{F_\mu})\bar{q}O^{\circledast}_{(e_\lambda,x_\alpha)}$, then $cl(O_{(e_\lambda,x_\alpha)})\bar{q}cl(O_{F_\mu})$.

Conversely, It is follows direct by the hypothesis. □

Theorem 3.10. *Let (X, T, E) be a $GFST-$space. Then (X, T, E) is a $GFS\ Q$ normal space if and only if for all $F_\mu \in T^c$ and for all $O_{F_\mu} \in N_q(F_\mu)$, there exist $O^*_{F_\mu}$ such that $cl(O^*_{F_\mu}) \sqsubseteq O_{F_\mu}$.*

Proof. It is analogous to that of Theorem 3.8. □

Theorem 3.11. *A $GFST-$space (X, T, E) is a $GFS\ Q$ normal space if and only if for every $F_\mu, G_\delta \in T^c$, with $F_\mu\bar{q}G_\delta$ implies there exist $O_{F_\mu}, O_{G_\delta} \in T$ such that $cl(O_{F_\mu})\bar{q}cl(O_{G_\delta})$.*

Proof. It is analogous to that of Theorem 3.9. □

5. Generalized fuzzy soft T_i-spaces, $i = 0, 1, 2, 3, 4$

In this section, we introduce the notion of generalized fuzzy soft quasi T_i-spaces ($GFS\ Q - T_i; i = 0, 1, 2, 3, 4$ for short) and the notion of generalized fuzzy soft hereditary property of the axioms $GFS\ Q - R_1$, $GFS\ Q - R_2$, $GFS\ Q$ regular and $GFS\ Q - T_j$ for $j = 0, 1, 2, 3$.

Definition 4.1. A $GFST-$space (X, T, E) is said to be:

(1) generalized fuzzy soft quasi T_0-space ($GFS\ Q - T_0-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ implies there exist $O_{(e_\lambda,x_\alpha)} \in N_q(e_\lambda, x_\alpha)$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}(e'_\gamma, y_\beta)$ or there exist $O_{(e'_\gamma,y_\beta)} \in N_q(e'_\gamma, y_\beta)$ such that $O_{(e'_\gamma,y_\beta)}\bar{q}(e_\lambda, x_\alpha)$.

(2)generalized fuzzy soft quasi T_1-space ($GFS\ Q - T_1-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ implies there exist $O_{(e_\lambda,x_\alpha)} \in N_q(e_\lambda, x_\alpha)$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}(e'_\gamma, y_\beta)$ and there exist $O_{(e'_\gamma,y_\beta)} \in N_q(e'_\gamma, y_\beta)$ such that $O_{(e'_\gamma,y_\beta)}\bar{q}(e_\lambda, x_\alpha)$.

(3) generalized fuzzy soft quasi T_2-space ($GFS\ Q - T_2-$space for short) if for every $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ implies there exist $O_{(e_\lambda,x_\alpha)} \in N_q(e_\lambda, x_\alpha)$ and $O_{(e'_\gamma,y_\beta)} \in N_q(e'_\gamma, y_\beta)$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$.

(4) generalized fuzzy soft quasi T_3-space ($GFS\ Q-T_3-$space for short) if $GFS\ Q$ *regular* and $GFS\ Q-T_1-$space.

(5) generalized fuzzy soft quasi T_4-space ($GFS\ Q-T_4-$space for short) if $GFS\ Q$ *normal* and $GFS\ Q-T_1-$space.

Theorem 4.2. *Let (X,T,E) be a $GFST-$space. Then (X,T,E) is a $GFS\ Q-T_0-$space if and only if for every $(e_\lambda,x_\alpha),(e'_\gamma,y_\beta)\in GFSP(X,E)$ with $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$ implies $(e_\lambda,x_\alpha)\bar{q}cl(e'_\gamma,y_\beta)$ or $cl(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$.*

Proof. Let (X,T,E) be a$GFS\ Q-T_0-$space and $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$. Then there exist $O_{(e_\lambda,x_\alpha)}$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}(e'_\gamma,y_\beta)$ or there exist $O_{(e'_\gamma,y_\beta)}$ such that $O_{(e'_\gamma,y_\beta)}\bar{q}(e_\lambda,x_\alpha)$. By (1) of proposition 3.3, we have $e_\lambda,x_\alpha)\bar{q}cl(e'_\gamma,y_\beta)$ or $(e'_\gamma,y_\beta)\bar{q}cl(e_\lambda,x_\alpha)$.

Conversely, let $(e_\lambda,x_\alpha)\bar{q}cl(e'_\gamma,y_\beta)$ or $cl(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$. Then $(e_\lambda,x_\alpha)\tilde{\in}[cl(e'_\gamma,y_\beta)]^c$ or $(e'_\gamma,y_\beta)\tilde{\in}[cl(e_\lambda,x_\alpha)]^c$. Take $O_{(e_\lambda,x_\alpha)}=[cl(e'_\gamma,y_\beta)]^c$ and $O_{(e'_\gamma,y_\beta)}=[cl(e_\lambda,x_\alpha)]^c$. Therefore, $(e_\lambda,x_\alpha)\bar{q}O_{(e'_\gamma,y_\beta)}$ or $(e'_\gamma,y_\beta)\bar{q}O_{(e_\lambda,x_\alpha)}$. Hence, (X,T,E) is a $GFS\ Q-T_0-$space. □

Theorem 4.3. *Let (X,T,E) be a $GFST-$space. The following are equivalent:*

(1) (X,T,E) is $GFS\ Q-T_1-$space.

(2) For every $(e_\lambda,x_\alpha),(e'_\gamma,y_\beta)\in GFSP(X,E)$ with $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$ implies $(e_\lambda,x_\alpha)\bar{q}cl(e'_\gamma,y_\beta)$ and $cl(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$.

(3) $cl(e_\lambda,x_\alpha)=(e_\lambda,x_\alpha)$ for every $(e_\lambda,x_\alpha)\in GFSP(X,E)$.

Proof. (1) $\Longrightarrow$(2): Let (X,T,E) be a $GFS\ Q-T_1-$space and $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$. Then there exist $O_{(e_\lambda,x_\alpha)}$ and $O_{(e'_\gamma,y_\beta)}$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}(e'_\gamma,y_\beta)$ and $O_{(e'_\gamma,y_\beta)}\bar{q}(e_\lambda,x_\alpha)$. By (1) of proposition 3.3, we have $(e_\lambda,x_\alpha)\bar{q}cl(e'_\gamma,y_\beta)$ and $(e'_\gamma,y_\beta)\bar{q}cl(e_\lambda,x_\alpha)$.

(2)$\Longrightarrow$(1): Follows directly from Proposition 3.3 (1).

(1)$\Longrightarrow$(3): Let $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$. Then there exist $O_{(e'_\gamma,y_\beta)}$ such that $O_{(e'_\gamma,y_\beta)}\bar{q}(e_\lambda,x_\alpha)$. This implies $O_{(e'_\gamma,y_\beta)}\sqsubseteq(e_\lambda,x_\alpha)^c$. Thus $(e_\lambda,x_\alpha)^c$ is a GFS open i.e., (e_λ,x_α) is a GFS closed. Hence, $cl(e_\lambda,x_\alpha)=(e_\lambda,x_\alpha)$ and this is true for every $(e_\lambda,x_\alpha)\in GFSP(X,E)$.

(3)$\Longrightarrow$(1): Let $cl(e_\lambda,x_\alpha)=(e_\lambda,x_\alpha)$ for every $(e_\lambda,x_\alpha)\in GFSP(X,E)$ and $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$. Then for every $(e_\lambda,x_\alpha),(e'_\gamma,y_\beta)\in T^c$. Since $(e'_\gamma,y_\beta)\bar{q}(e'_\gamma,y_\beta)^c=O_{(e_\lambda,x_\alpha)}$ and $(e_\lambda,x_\alpha)\bar{q}(e_\lambda,x_\alpha)^c=O_{(e'_\gamma,y_\beta)}$. Hence, (X,T,E) is $GFS\ Q-T_1-$space. □

Theorem 4.4. *Let (X,T,E) be a $GFST-$space. If (X,T,E) is a $GFS\ Q-T_2-$space, then $(e_\lambda,x_\alpha)=\sqcap\{cl(O_{(e_\lambda,x_\alpha)}):O_{(e_\lambda,x_\alpha)}\in N_q(e_\lambda,x_\alpha)\}$ for all $(e_\lambda,x_\alpha)\in GFSP(X,E)$.*

Proof. Let (X,T,E) be a $GFS\ Q-T_2-$space and $(e_\lambda,x_\alpha)\in GFSP(X,E)$. Then for any $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$ there exist $O_{(e_\lambda,x_\alpha)}$ and $O_{(e'_\gamma,y_\beta)}$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$. By (1) of proposition 3.3, we have $(e'_\gamma,y_\beta)\bar{q}cl(O_{(e_\lambda,x_\alpha)})$ and so $(e'_\gamma,y_\beta)\bar{q}\sqcap\{cl(O_{(e_\lambda,x_\alpha)}):O_{(e_\lambda,x_\alpha)}\in N_q(e_\lambda,x_\alpha)\}$.

By (5) of proposition 3.2, $\sqcap\{cl(O_{(e_\lambda,x_\alpha)}):O_{(e_\lambda,x_\alpha)}\in N_q(e_\lambda,x_\alpha)\}\sqsubseteq(e_\lambda,x_\alpha)$. But $(e_\lambda,x_\alpha)\tilde{\in}\sqcap\{cl(O_{(e_\lambda,x_\alpha)}):O_{(e_\lambda,x_\alpha)}\in N_q(e_\lambda,x_\alpha)\}$. This complete the proof. □

Theorem 4.5. *The following implication holds:*

$$GFS\ Q-T_4 \Longrightarrow GFS\ Q-T_3 \Longrightarrow GFS\ Q-T_2 \Longrightarrow GFST_1 \Longrightarrow GFS\ Q-T_0$$

Proof. (i) $GFS\ Q-T_4 \Longrightarrow GFS\ Q-T_3$: Let (X,T,E) be a $GFS\ Q-T_4-$space and $(e_\lambda,x_\alpha)\bar{q}F_\mu$ where $F_\mu \in T^c$. Then $cl(e_\lambda,x_\alpha)=(e_\lambda,x_\alpha)$ implies $cl(e_\lambda,x_\alpha)\bar{q}F_\mu$. Since (X,T,E) is a $GFS\ Q$ normal space, then there exist $O_{cl(e_\lambda,x_\alpha)}$ and $O_{F_\mu} \in T$ such that $O_{cl(e_\lambda,x_\alpha)}\bar{q}O_{F_\mu}$. Now put $O_{(e_\lambda,x_\alpha)} = O_{cl(e_\lambda,x_\alpha)}$, then $O_{(e_\lambda,x_\alpha)}\bar{q}O_{F_\mu}$. Hence (X,T,E) is a $GFS\ Q-T_3-$space.

(ii) $GFS\ Q-T_3 \Longrightarrow GFS\ Q-T_2$: Let (X,T,E) be a $GFS\ Q-T_3-$space and $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$ implies $cl(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$ and $cl(e_\lambda,x_\alpha)\in T^c$. Since (X,T,E) is a $GFS\ Q$ regular space, then there exist $O_{cl(e_\lambda,x_\alpha)}$ and $O_{(e'_\gamma,y_\beta)} \in T$ such that $O_{cl(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$. Now put $O_{(e_\lambda,x_\alpha)} = O_{cl(e_\lambda,x_\alpha)}$. Then $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$. Hence (X,T,E) is a $GFS\ Q-T_2-$space.

(iii) $GFS\ Q-T_2 \Longrightarrow GFS\ Q-T_1$ and $GFS\ Q-T_1 \Longrightarrow GFS\ Q-T_0$ are immediate. □

Corollary 4.6. *The following implication holds:*

$$\begin{array}{ccccccc} GFS\ Q-T_4 \Longrightarrow & & GFS\ Q-T_3 \Longrightarrow & & GFS\ Q-T_2 \Longrightarrow GFS\ Q-T_1 \Longrightarrow GFS\ Q-T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \qquad\qquad \Downarrow \\ GFS\ Q\ normal \wedge GFS\ Q-R_0 & \Longrightarrow & GFS\ Q\ regular & \Longrightarrow & GFS\ Q-R_1 \Longrightarrow GFS\ Q-R_0 \end{array}$$

Proof. (i) The implications $GFS\ Q-T_4 \Longrightarrow GFS\ Q-T_3 \Longrightarrow GFS\ Q-T_2 \Longrightarrow GFS\ Q-T_1 \Longrightarrow GFS\ Q-T_0$ and $GFS\ Q\ normal \wedge GFS\ Q-R_0 \Longrightarrow GFS\ Q\ regular \Longrightarrow GFS\ Q-R_1 \Longrightarrow GFS\ Q-R_0$ are hold by Theorems 4.5, 3.6.

(ii) $GFS\ Q-T_3 \Longrightarrow GFS\ Q$ regular is obvious by from definition 4.1.

(iii) $GFS\ Q-T_2 \Longrightarrow GFS\ Q-R_1$: Let (X,T,E) be a $GFS\ Q-T_2-$space and $(e_\lambda,x_\alpha)\bar{q}cl(e'_\gamma,y_\beta)$ implies that $(e_\lambda,x_\alpha)\bar{q}(e'_\gamma,y_\beta)$. Since (X,T,E) is a $GFS\ Q-T_2-$space, then there exist $O_{(e_\lambda,x_\alpha)} \in N_q(e_\lambda,x_\alpha)$ and $O_{(e'_\gamma,y_\beta)} \in N_q(e'_\gamma,y_\beta)$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$. Hence (X,T,E) is a $GFS\ Q-R_1-$space.

(iv) $GFS\ Q-T_1 \Longrightarrow GFS\ Q-R_0$: Let (X,T,E) be a $GFS\ Q-T_1-$space. By (3) of Theorem 4.3, then $cl(e_\lambda,x_\alpha) = (e_\lambda,x_\alpha)$ for every $(e_\lambda,x_\alpha)\in FSP(X,E)$. Since $(e_\lambda,x_\alpha)\widetilde{\in}O_{(e_\lambda,x_\alpha)}$ i.e. $O_{(e_\lambda,x_\alpha)}\in N_q(e_\lambda,x_\alpha)$, Then $cl(e_\lambda,x_\alpha)\sqsubseteq O_{(e_\lambda,x_\alpha)}\forall O_{(e_\lambda,x_\alpha)}\in N_q(e_\lambda,x_\alpha)$. By (3) of Theorem 3.5, we have (X,T,E) is a $GFS\ Q-R_0-$space.

(v) $GFS\ Q-T_4 \Longrightarrow GFS\ Q\ normal \wedge GFS\ Q-R_0$: Let (X,T,E)be a $GFS\ Q-T_4-$space. Then (X,T,E) is a $GFS\ Q$ normal and $GFS\ Q-T_1-$space. Hence,(X,T,E) is a $GFS\ Q$ normal and $GFS\ Q-R_0-$space. □

Definition 4.7. *A GFSS G_δ is said to be GFS quasi-coincident with H_ν with respect to a GFSS F_μ, denoted by $G_\delta q_{F_\mu} H_\nu$, if there exists $x,e\in S(F_\mu)$ such that $G(e)(x)+H(e)(x) > F(e)(x)$ and $\delta(e)+\nu(e) > \mu(e)$. In particular, $(e_\lambda,x_\alpha)q_{F_\mu}G_\delta$ if $\alpha + G(e)(x) > F(e)(x)$ and $\lambda+\delta(e) > \mu(e)$.*

Definition 4.8. *The property P is said to be a generalized fuzzy soft hereditary property (GFS hereditary property for short) if (X,T,E) is a GFST-space has the GFS property P, then every GFS subspace has the P.*

Now, the following theorems shows that the axioms GFS $Q-R_0$, GFS $Q-R_1$, GFS Q regular and GFS $Q-T_j$ for $j=0,1,2,3$ are GFS hereditary property.

Theorem 4.9. *Let (X,T,E) be a GFST$-$space over (X,E) and $G_\delta$be GFS subset of $\widetilde{1}_\triangle$. If (X,T,E) is a GFS $Q-T_j-$space, then $(G_\delta, T_{G_\delta}, E)$ is a GFS $Q-T_j-$space for $j=0,1,2,3$.*

Proof. *As a sample, we will prove the case $j=0$. Let (X,T,E) be a GFS $Q-T_0-$space, $(e_\lambda,x_\alpha),(e'_\gamma,y_\beta)\widetilde{\in}G_\delta$ with $(e_\lambda,x_\alpha)\bar{q}_{G_\delta}(e'_\gamma,y_\beta)$. Then $(e_\lambda,x_\alpha)\bar{q}_{\widetilde{1}_\triangle}(e'_\gamma,y_\beta)$. Since (X,T,E) is a GFS $Q-T_0-$space, then there exists $O_{(e_\lambda,x_\alpha)}\in T$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}_{\widetilde{1}_\triangle}(e'_\gamma,y_\beta)$ or there exists $O_{(e'_\gamma,y_\beta)}\in T$ such that $O_{(e'_\gamma,y_\beta)}\bar{q}_{\widetilde{1}_\triangle}(e_\lambda,x_\alpha)$. Take $O^*_{(e_\lambda,x_\alpha)}=O_{(e_\lambda,x_\alpha)}\sqcap G_\delta$ or $O^*_{(e'_\gamma,y_\beta)}=O_{(e'_\gamma,y_\beta)}\sqcap G_\delta$. Therefore, $O^*_{(e_\lambda,x_\alpha)}\bar{q}_{G_\delta}(e'_\gamma,y_\beta)$ or $O^*_{(e'_\gamma,y_\beta)}\bar{q}_{G_\delta}(e_\lambda,x_\alpha)$. Hence, $(G_\delta,T_{G_\delta},E)$ is a GFS $Q-T_0-$space.* □

Theorem 4.10. *Let (X,T,E) be a GFST$-$space over (X,E) and G_δ be GFS subset of $\widetilde{1}_\triangle$. If (X,T,E) is a GFS $Q-R_0-$space, GFS $Q-R_1-$space and GFS Q regular space then $(G_\delta,T_{G_\delta},E)$ is GFS $Q-R_0-$space, GFS $Q-R_1-$space and GFS Q regular space.*

Proof. *As a sample, we will prove the GFS Q regular space. Let (X,T,E) be a GFS Q regular space., $(e_\lambda,x_\alpha)\widetilde{\in}G_\delta$ and H_ν be a GFS closed subset of G_δ with $(e_\lambda,x_\alpha)\bar{q}_{G_\delta}H_\nu$. Then $(e_\lambda,x_\alpha)\bar{q}_{G_\delta}cl_{T_{G_\delta}}(H_\nu)$. Since $cl_{T_{G_\delta}}(H_\nu)=cl_T(H_\nu)\sqcap G_\delta$. then $(e_\lambda,x_\alpha)\bar{q}_{G_\delta}[cl_T(H_\nu)\sqcap G_\delta]$. This implies $\alpha+min\{cl_T(H)(e)(x),G(e)(x)\}\le G(e)(x),\lambda+min\{cl_T(\nu)(e),\delta(e)\}\le\delta(e)$.*

Now, If $G(e)(x)=0,\delta(x)=0$, then $\alpha=0,\lambda=0$.

If $G(e)(x)\neq 0$, $\delta(x)\neq 0$, then $G(e)(x)=\widetilde{1}(e)(x),\delta(x)=\triangle(e)$ and so $\alpha+cl_T(H)(e)(x)\le\widetilde{1}(e)(x)=1,\lambda+cl_T(\nu)(e)\le\triangle(e)=1$. Therefore, $(e_\lambda,x_\alpha)\bar{q}_{\widetilde{1}_\triangle}cl_T(H_\nu)$. Since (X,T,E) is a GFS Q regular space, then there exist $O_{(e_\lambda,x_\alpha)}$and $O_{cl_T(H_\nu)}\in T$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}_{\widetilde{1}_\triangle}O_{cl_T(H_\nu)}$. Take $O^*_{(e_\lambda,x_\alpha)}=O_{(e_\lambda,x_\alpha)}\sqcap G_\delta, O^*_{cl_T(H_\nu)}=O_{cl_T(H_\nu)}\sqcap G_\delta\in T_{G_\delta}$. Hence, $O^*_{(e_\lambda,x_\alpha)}\bar{q}_{G_\delta}O^*_{cl_T(H_\nu)}$, and so $(G_\delta,T_{G_\delta},E)$ is a GFS Q regular space. □

Theorem 4.11. *Let (X,T,E) be a GFST$-$space over (X,E) and G_δ be GFS closed subset of $\widetilde{1}_\triangle$. If (X,T,E) is a GFS Q normal space, then $(G_\delta,T_{G_\delta},E)$ is a GFS Q normal space.*

Proof. *Let (X,T,E) be a GFS Q normal space, $G_\delta\in T^c$. Suppose H_ν and K_γ are GFS closed subsets of G_δ. Then H_ν and K_γ are GFS closed subsets of $\widetilde{1}_\triangle$. Since (X,T,E) is a GFS Q normal space, then there exist O_{H_ν} and $O_{K_\gamma}\in T$ such that $O_{H_\nu}\bar{q}_{\widetilde{1}_\triangle}O_{K_\gamma}$. Take $O^*_{H_\nu}=O_{H_\nu}\sqcap G_\delta\in T_{G_\delta}$ and $O^*_{K_\gamma}=O_{K_\gamma}\sqcap G_\delta\in T_{G_\delta}$. Hence, $O^*_{H_\nu}\bar{q}_{G_\delta}O^*_{K_\gamma}$, and so $(G_\delta,T_{G_\delta},E)$ is a GFS Q normal space.* □

5. Conclusion

In 2018, we [12] have formulated the concept of the separation axioms T_i ($i=0,1,2,3,4$) in generalized fuzzy soft topological spaces as an extended of the separation axioms in fuzzy soft topological spaces by using generalized fuzzy soft open sets and generalized fuzzy soft neighborhood system. In this paper, we use the notions of generalized fuzzy soft quasi-coincident relation and generalized fuzzy soft quasi -neighborhood system [19] to define a quasi separation axioms in generalized fuzzy soft fuzzy topological, which is an extension of Kandil's definition of quasi separation axioms in the case fuzzy soft fuzzy topological [10].

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